Behavior of the Family of Trinomial  

\textbf{Arcs }G(p, k, r, n) \textbf{ when } 0 < \alpha < 1

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Abstract

The family of trinomial arcs \( G(p, k, r, n) \) is the set of roots of equation  
\[ z^n = \alpha z^k + (1 - \alpha), \]
where \( z = \rho e^{i\theta} \), \( n \) and \( k \) are two integers such that \( k = 1, 2, ..., n-1 \), \( p \) is an integer, \( r \) is a nonzero integer and \( \alpha \) is a real number between 0 and 1. From the equation of trajectories of trinomial roots  
\[ \rho^{n-k} \sin n\theta - \rho^n \sin(n - k)\theta = \sin k\theta, \]
we will study the behavior of this family of arcs. We will prove that \( \rho \) changes monotonically with respect to \( \theta \) and that \( \rho(\theta) \) is an increasing function, for each trinomial arc \( G(p, k, r, n) \).

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1 Introduction

Prolonging the works of Nekrassof [7], Kempner [6] and Biernacki [1], Fell [5] has established a large description of the trajectories of roots of the trinomial equation

\[ z^n = \alpha \, z^k + (1 - \alpha) \quad (1) \]

where \( z \) is a complex number, \( n \) is an integer larger than one, \( k = 1, 2, ..., n-1 \) and \( \alpha \) is a real number. These trajectories, called \textit{trinomial arcs}, are continuous arcs, corresponding to a number \( \alpha \) which is whether between 0
and 1, or between 1 and $+\infty$, or also between $-\infty$ and 0. Each of these arcs can be expressed in polar coordinates $(\rho, \theta)$ by a function $\rho(\theta)$. However, Fell [5] could not establish the monotonicity of $\rho$ as a function of $\theta$. Otherwise, in [4], Dubuc and Zaoui have studied the quasi-convexity of some particular trinomial arcs corresponding to the case $0 < \alpha < 1$. In this paper, we will restrict our attention to one family of trinomial arcs, solutions of equation (1) with $0 < \alpha < 1$, denoted by $G(p, k, r, n)$, where $p$, $k$, $r$ and $n$ satisfy some conditions. In order to prove that $\rho(\theta)$ is a monotonic function for these arcs, we start by formulating and defining this family of curves $G(p, k, r, n)$. For these arcs, $\rho(\theta)$ will be shown to be a derivable function. Next, three important intermediate results are proved. Finally, we obtain that $\rho(\theta)$ is an increasing function for each trinomial arc $G(p, k, r, n).

2 The trinomial equation in polar coordinates

If $z = \rho e^{i\theta}$ in the trinomial equation (1), we get $\rho^n e^{in\theta} = \alpha \rho^k e^{ik\theta} + (1 - \alpha)$. Separating real and imaginary parts, one has $\rho^n \sin n\theta = \alpha \rho^k \sin k\theta$ and $\rho^n \cos n\theta = \alpha \rho^k \cos k\theta + (1 - \alpha)$. So, when $\theta \neq l\pi/n$, where $l$ is an integer, one gets

$$\rho^{n-k} = \alpha \frac{\sin k\theta}{\sin n\theta}$$

(2)

Now, divide the equation (1) by $z^n$ and consider the imaginary part. When $\alpha \neq 0$ and $\theta \neq l\pi/(n-k)$, where $l$ is an integer, we obtain that

$$\rho^k = (1 - 1/\alpha) \frac{\sin n\theta}{\sin(n-k)\theta}$$

(3)

Therefore, we have the $\alpha$-free equation for the trajectories of roots of (1):

$$\rho^{n-k} \sin n\theta - \rho^n \sin(n-k)\theta = \sin k\theta$$

(4)

As $0 \leq \alpha \leq 1$, it follows that

$$\text{sign} (\sin n\theta) = \text{sign} (\sin k\theta) = -\text{sign} (\sin(n-k)\theta)$$

(5)

Definition 2.1 An angle $\theta$ which fulfills (5) will be called a $(n, k)$-feasible angle for the equation (1) with $0 \leq \alpha \leq 1$.

Remark 2.2 The upper and lower half-planes are symmetrical. So, we will restrict our study to the upper half-plane.

On the other hand, the trajectories of roots of equation (1) with $0 \leq \alpha \leq 1$ are inside the unit disk, in view of the next result of [3]:
Lemma 2.3 For any \((n, k)\)-feasible angle \(\theta\) for the equation (1) with \(0 \leq \alpha \leq 1\), the function of \(\rho\), 
\[- \rho^n \{\sin (n - k) \theta / \sin k \theta\} + \rho^{n-k} \{\sin n \theta / \sin k \theta\} - 1,\]
is increasing and vanishes for one and only one positive value of \(\rho\), which is not larger than 1.

In fact, Fell has studied in [5], the trinomial equation
\[\lambda z^n + (1 - \lambda) z^k - 1 = 0,\tag{6}\]
where \(z\) is a complex number, \(n\) and \(k\) are two integers such that \(k = 1, 2, ..., n - 1\) and \(\lambda\) is a real number. Substituting into equation (6) the expression given for \(z^n\) by equation (1) yields
\[\left(z^k - 1\right) \left[1 - \lambda (1 - \alpha)\right] = 0.\]
So, \(z^k = 1\) or \(\lambda (1 - \alpha) = 1\). Because \(z\) is a complex number, then \(\alpha = 1 - 1/\lambda\). Hence, so as to pass from (1) to (6), we put \(\alpha = 1 - 1/\lambda\). From this equality stems easily that the case \(0 \leq \alpha \leq 1\) of equation (1) corresponds to the case \(1 \leq \lambda < +\infty\) of equation (6).

Remark 2.4 The cases \(\alpha = 0\) and \(\alpha = 1\) are two particular cases for the trinomial equation (1). In fact, when \(\alpha = 0\), equation (1) becomes \(z^n = 1\). Then, its zeros are the \(n\)th roots of unity. In the case \(\alpha = 1\), (1) becomes \(z^n = z^k\), i.e. \(z^k \left[z^{n-k} - 1\right] = 0\). So, the \(n\) roots of (1) are exactly the \((n - k)\)th roots of unity, which are simple roots and 0; a root of multiplicity \(k\).

3 Trinomial arcs \(G(p, k, r, n)\)

When \(\alpha = 0\), the equation (1) has \(n\) solutions; the \(n\)th roots of unity. When \(\alpha\) moves from 0 to 1, the \(n\) trajectories of the \(n\) roots are continuous arcs inside the unit disk. According to Fell [5], there exist \(k\) trajectories of these arcs going to 0, of which the tangents are a line segments joining 0 and an \(k\)th root of \(-1\). These \(k\) trajectories are described in [5] in the following manner. Let
\[C = \{n\text{th roots of unity}\},\]
\[D = \{(n - k)\text{th roots of unity}\}\]
and
\[E = \{k\text{th roots of -1}\}\]

Let \(\gamma \in C\) and \(\delta\) be the unique nearest neighbor of \(\gamma\) such that \(\delta \in E\) and \(\delta \notin D\). When \(\alpha\) moves from 0 to 1, the trajectory starting at \(\gamma\) when \(\alpha = 0\) is a continuous arc such that \(\rho\) changes from 1 to 0. This trajectory is tangent to the line segment \(\theta = \text{arg}(\delta)\). Fell has showed in [5] that, in the case \(\delta \in E\), \(\delta \notin D\) and \(0 < \alpha < 1\), the feasible angles \(\theta\) belong to intervals of length less than or equal to \(\pi/n\) and bounded by \(\text{arg}(\delta)\) and \(\text{arg}(\gamma)\), such that \(\delta\) is an \(k\)th root of \(-1\) and \(\gamma\) is an \(n\)th root of unity.

Thus, we will consider and restrict our attention, in this paper, to the trinomial arcs such that the feasible angles \(\theta\) belong to an interval of the form
\[(2p + 1)\pi/k, \ 2\pi r/n\], where \(p\) and \(r\) are two integers. In fact, \(\delta\) such that \(\text{arg}(\delta) = (2p + 1)\pi/k\) is an \(k\)th root of \(-1\) and \(\gamma\) such that \(\text{arg}(\gamma) = 2\pi r/n\) is an \(n\)th root of unity. These trinomial curves will be denoted by \(G(p,k,r,n)\).

When \(n = 2\), the trajectories of roots of equation (1) with \(0 < \alpha < 1\) are linear, then we define the family of trinomial arcs \(G(p,k,r,n)\) as follows:

**Definition 3.1** If \(n\) is an integer greater than or equal to 3, so \(G(p,k,r,n)\) is the set of roots of equation (1) with \(0 < \alpha < 1\) and the feasible angles belong to \([((2p + 1)\pi/k, \ 2\pi r/n)\], where \(p\) is an integer, \(r\) is a nonzero integer verifying \(r \geq p + 1\) and \(k\) is an integer such that \((2p + 1)\pi/k < n\theta < (2p + 1)_n/(2r - 1)\).

These arcs \(G(p,k,r,n)\) (see the figure below) exist in view of the lemma:

**Lemma 3.2** If \(n\) is an integer greater than or equal to 3, \(k\) is an integer such that \(1 \leq k < n\) and \(0 < \alpha < 1\), then in the trinomial equation (1) with \((2p + 1)\pi/k < n\theta < (2p + 1)_n/(2r - 1)\), where \(p\) is an integer, \(r\) is a nonzero integer verifying \(r \geq p + 1\), any angle of \([((2p + 1)\pi/k, \ 2\pi r/n)\] is feasible.

**Proof.** Let \(k\) be the integer satisfying \((2p + 1)n/2r < k < (2p + 1)n/(2r - 1)\). Notice that because \(0 < k < n\), the two integers \(p\) and \(r\) verify the condition \((2p + 1)/(2r - 1) \leq 1\), namely \(r \geq p + 1\). Let be \((2p + 1)\pi/k < \theta < 2\pi r/n\). We have \((2p + 1)\pi n/k < n\theta < 2\pi r\). Because \(k < (2p + 1)n/(2r - 1)\), it follows that \((2r - 1)\pi < (2p + 1)\pi n/k\) and that \(\sin n\theta < 0\). On the other hand, we have \((2p + 1)\pi < k\theta < 2\pi r k/n\). The fact that \(r \geq p + 1\) implies that \(k < (2p + 1)n/(2r - 1) \leq (p + 1)n/r\). So, \(2\pi r k/n < 2(p + 1)\pi\) and \(\sin k\theta < 0\). Lastly, we have \((2p + 1)\pi (n/k - 1) < (n - k)\theta < 2\pi r (1 - k/n)\). As \((2p + 1)n/2r < k < (2p + 1)n/(2r - 1)\), so \(2(r - p - 1)\pi < (2p + 1)\pi (n/k - 1)\) and \(2\pi r (1 - k/n) < [2(r - p) - 1]\pi\). Then \(\sin (n - k)\theta > 0\). Therefore, the conditions (5) are fulfilled.

Trinomial arcs \(G(p,k,r,n)\) inside the upper half unit disk

**Remark 3.3** According to the proof of Lemma 3.2, we have \(\sin n\theta < 0\), \(\sin k\theta < 0\) and \(\sin (n - k)\theta > 0\) for any \(\theta\) in the interval \([((2p + 1)\pi/k, \ 2\pi r/n)\].
4 Monotonicity of the function \( \rho(\theta) \) for the arcs \( G(p, k, r, n) \)

We start this section by proving that \( d\rho/d\theta \) exists and it is well-defined.

**Proposition 4.1** The function \( \rho(\theta) \) is derivable for each arc \( G(p, k, r, n) \).

**Proof.** Let \( G(p, k, r, n) \) be a trinomial arc. From (3), we have \( \rho^k(\theta) = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta \). By Remark 3.3, the feasible angles \( \theta \) verify \( \sin n\theta < 0 \) and \( \sin(n - k)\theta > 0 \). If we define \( h(\theta) = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta \) and since \( 0 < \alpha < 1 \), the denominator of \( h(\theta) \) is never zero. The function \( h(\theta) \) is then well-defined. Moreover, \( h(\theta) \) is derivable and positive. Then, \( \rho(\theta) = [h(\theta)]^{1/k} \) is a derivable function. Hence, its derivative \( d\rho/d\theta \) exists and it is well-defined. So, we achieve the proof.

In what follows, our main interest is to prove that for each trinomial arc \( G(p, k, r, n) \), the function \( \rho(\theta) \) is monotonic, i.e. that \( d\rho/d\theta \neq 0 \). We begin by differentiating both sides of equation (4) with respect to \( \theta \), we obtain so

\[
\left[ (n - k) \rho^{n-k-1} \sin n\theta - n \rho^{n-1} \sin(n - k)\theta \right] \frac{d\rho}{d\theta} = k \cos k\theta + (n - k) \rho^n \cos(n - k)\theta - n \rho^{n-k} \cos n\theta.
\]

Supposing that \( d\rho/d\theta = 0 \) and considering the \( \alpha \)-free equation (4) of trajectories of roots, \( \rho^{n-k} \) and \( \rho^n \) will be considered as solutions of the system:

\[
\begin{align*}
    k \cos k\theta + (n - k) \rho^n \cos(n - k)\theta - n \rho^{n-k} \cos n\theta &= 0 \\
    \rho^{n-k} \sin n\theta - \rho^n \sin(n - k)\theta - \sin k\theta &= 0
\end{align*}
\]

which is equivalent to the next system:

\[
\begin{align*}
    R(\theta) \cdot \rho^{n-k} &= N_1(\theta) \\
    R(\theta) \cdot \rho^n &= N_2(\theta)
\end{align*}
\]

with

\[
\begin{align*}
    R(\theta) &= (n - k) \sin k\theta - k \cos n\theta \sin(n - k)\theta \\
    N_1(\theta) &= (n - k) \sin n\theta - n \sin(n - k)\theta \cos k\theta \\
    N_2(\theta) &= (n - k) \sin k\theta \cos n\theta - k \sin(n - k)\theta.
\end{align*}
\]

Remarking that \( N_2(\theta) = (n - k) \sin n\theta \cos k\theta - n \sin(n - k)\theta \) and doing the difference of the two equations of (7), we arrive at the equality:

\[
R(\theta) [\rho^n - \rho^{n-k}] = U(\theta) [1 - \cos k\theta]
\]

where

\[
U(\theta) = -[n \sin(n - k)\theta + (n - k) \sin n\theta].
\]
Now, the question is to show that the hypothesis $d\rho/d\theta = 0$ is impossible for the trinomial arcs $G(p, k, r, n)$. Thus, we need the three following lemmas.

Lemma 4.2 For any feasible angle $\theta$ in the interval $] (2p+1)\pi/k, 2\pi r/n [$, we have $U(\theta) = -[n \sin(n - k) \theta + (n - k) \sin n\theta] < 0$.

Proof. Let $\theta$ be an angle in $] (2p+1)\pi/k, 2\pi r/n [$, The zeros of the derivative $U'(\theta) = -n(n - k)[\cos(n - k) \theta + \cos n\theta]$ are of the form $\theta = (2l-1)\pi/k$ or of the form $\theta = (2l+1)\pi/(2n - k)$ where $l$ is an integer. However, $(2l-1)\pi/k \in ](2p+1)\pi/k, 2\pi r/n [$ if and only if $(p+1) < l < \pi k/n + 1/2$. Because $k < (2p+1)n/(2r-1)$ and $r \geq p+1$, we get $rk/n + 1/2 < p + 3/2$. So, $p + 1 < l < r + 3/2$, which cannot occur as $l$ is an integer. On the other side, $(2l+1)\pi/(2n - k) \in ](2p+1)\pi/k, 2\pi r/n [$ if and only if $(2p+1)(n/k - 1/2) - 1/2 < l < 2r(1 - k/2n) - 1/2$. Since $k < (2p+1)n/(2r-1)$, so $2r - p - 2 < (2p+1)(n/k - 1/2) - 1/2$ and since $(2p+1)n/2r < k$, so $2r(1 - k/2n) - 1/2 < 2r - p - 1$. Then, the integer $l$ is such that $2r - p - 2 < l < 2r - p - 1$, which is not possible. We deduce that $U(\theta)$ is a monotonic function on the interval $] (2p+1)\pi/k, 2\pi r/n [$, Moreover, we have $U((2p+1)\pi/k) < 0$ and $U(2\pi r/n) < 0$. Therefore, $U(\theta) < 0$ for any angle $\theta$ in $] (2p+1)\pi/k, 2\pi r/n [$.

In order to prove the two next lemmas, we will need the remark below:

Remark 4.3 For any feasible angle $\theta$ in $] (2p+1)\pi/k, 2\pi r/n [$, the sign of $\cos n\theta$ is determined as follows:

Case i): $(2p+1)n/2r < k \leq 2(2p+1)n/(4r - 1)$. In this case, $\cos n\theta > 0$ for any $\theta$ in the interval $] (2p+1)\pi/k, 2\pi r/n [$.

Case ii): $2(2p+1)n/(4r - 1) < k < (2p+1)n/(2r-1)$. We have $\cos n\theta > 0$ on $] (2p+1)\pi/k, 2\pi r/n [$ if and only if $\theta = (4r - 1)\pi/2n$. Moreover, we have $\cos n\theta < 0$ for $\theta < (4r - 1)\pi/2n$ and $\cos n\theta > 0$ for $\theta > (4r - 1)\pi/2n$.

Lemma 4.4 For any integer $k$ such that $(2p+1)n/2r < k < (2p+1)n/(2r-1)$, we have $N_2(\theta) = [(n - k) \sin k\theta \cos n\theta - k \sin(n - k) \theta] < 0$, for any angle $\theta$ in the interval $] (2p+1)\pi/k, 2\pi r/n [$.

Proof. From Remark 3.3, we have $\sin k\theta < 0$ and $\sin(n - k) \theta > 0$ for any $\theta$ in $] (2p+1)\pi/k, 2\pi r/n [$, So, by remark above, in Case i), we have $N_2(\theta) < 0$ for any angle $\theta$ in $] (2p+1)\pi/k, 2\pi r/n [$, and in Case ii), we distinguish the two following subcases:

Subcase 1): $\theta$ belongs to $] (2p+1)\pi/k, (4r - 1)\pi/2n [$, Remarking that $N_2(\theta)$ can be expressed as $N_2(\theta) = n \sin k\theta \cos n\theta - k \sin n\theta \cos k\theta$, we will consider the function $M_2(\theta) = N_2(\theta) / \cos n\theta \cos k\theta = n \tan k\theta - k \tan n\theta$, we have $\cos n\theta < 0$ for any $\theta$ in $] (2p+1)\pi/k, (4r - 1)\pi/2n [$, and in addition, we have
in the interval $(2p+1)\pi/k < 2(2p+1)/(2r-1)$. Since $r \geq p+1$, we get $k/n < (2p+1)/(2r-1) \leq (2p+3)/(2r-1/2)$. It follows that $(2p+1)\pi < k\theta < (2p+3)/(2)\pi$ and that $\cos k\theta < 0$. Therefore, $N_2(\theta)$ has the same sign as $M_2(\theta)$, which is derivable with $M'_2(\theta) = nk[\tan^2 k\theta - \tan^2 n\theta]$. Because $\tan n\theta > 0$ and $\tan k\theta > 0$, the zeros of this derivative are solutions of the equation $\tan n\theta = \tan k\theta$. The unique solution of this equation is of the form $R$ expressed as $\tan(n\theta) = R$ if and only if $(2p+1)n/k < k\theta < (2p+3)n/2$. Therefore, $N_2(\theta)$ goes to $-\infty$ as $\theta$ tends on the left to $(4r-1)\pi/2n$. Then, we deduce that $M_2(\theta) < 0$ and that $N_2(\theta) < 0$ for any $\theta$ in $]2(2p+1)\pi/k, (4r-1)\pi/2n[$.

Subcase 2: $\theta$ belongs to $](4r-1)\pi/2n, 2\pi r/n]$. In this situation, we have $\tan n\theta < 0$. Therefore, for any integer $k$ such that $(2p+1)n/(2r-1) < k < (2p+1)n/(2r-1)$, we have $N_2(\theta) < 0$ for any feasible angle $\theta$ in $]2(2p+1)\pi/k, 2\pi r/n[$.

**Lemma 4.5** For any integer $k$ such that $(2p+1)n/(2r-1) < k < (2p+1)n/(2r-1)$, the sign of $R(\theta) = (n-k)\sin k\theta - k\cos n\theta \sin(n-k)\theta$ is as follows:

- If $(2p+1)n/(2r-1) < k < 2(2p+1)n/(4r-1)$, so $R(\theta) < 0$ for any angle $\theta$ in the interval $]2(2p+1)\pi/k, 2\pi r/n[$.

- If $2(2p+1)n/(4r-1) < k < (2p+1)n/(2r-1)$, there exists an angle $\theta_c$ in $]2(2p+1)\pi/k, 2\pi r/n[$ such that $R(\theta_c) = 0$. Moreover, $R(\theta) > 0$ for $\theta < \theta_c$ and $R(\theta) < 0$ for $\theta > \theta_c$.

**Proof.** According to Remark 4.3, we have in the case i) that $R(\theta) < 0$ for any angle $\theta$ in $]2(2p+1)\pi/k, 2\pi r/n[$. But in the case ii), two subcases are possible:

Subcase 1: $\theta$ belongs to $]2(2p+1)\pi/k, (4r-1)\pi/2n[$. Because $R(\theta)$ can be expressed as $R(\theta) = (n-k)\sin n\theta \cos(n-k)\theta - n\sin(n-k)\theta \cos n\theta$, we define the function $K(\theta) = R(\theta)/\cos n\theta \cos(n-k)\theta = (n-k)\tan n\theta - n\tan(n-k)\theta$. In this case, we have $\cos n\theta < 0$. Moreover, we have $(2p+1)\pi(n-k)/k < (n-k)\theta < (2r-1)\pi/(1-k/n)$. As $k < (2p+1)n/(2r-1)$, so $2(2p+1)n/(2r-1) < k < (2p+1)\pi/(2r-1)$ and because $2(2p+1)n/(4r-1) < k$, $(2p+1)n/(2r-1) < (2r-1)\pi/(1-k/n) < 2(2r-1)\pi/3$. This implies that $\tan(n-k)\theta > 0$. Then, the sign of $R(\theta)$ is opposite to that of $K(\theta)$. The zeros of $K'(\theta) = n(n-k)[\tan^2 n\theta - \tan^2(n-k)\theta]$ are those of the equation $\tan n\theta = \tan(n-k)\theta$, because $\tan n\theta > 0$ and $\tan(n-k)\theta > 0$. Hence, the unique solution of this equation is of the form $\theta = l\pi/k$ where $l$ is an integer. However, $l\pi/k \in ]2(2p+1)\pi/k, (4r-1)\pi/2n[$ is equivalent to $(2p+1) \frac{l\pi}{k} < (2p+1)/(2r-1)$. Since $k < (2p+1)n/(2r-1)$ and $r \geq p+1$, so...
(2r−1/2)k/n < (2p+3/2). From this, we have (2p+1) < l < (2p+3/2), which is impossible because l is an integer. So, we deduce that K′(θ) ≠ 0. On the other hand, K((2p+1)π/k) < 0 and K(θ) goes to +∞ as θ tends on the left to (4r−1)π/2n. Then, there exists θc in the interval ](2p+1)π/k, (4r−1)π/2n[ such that R(θc) = 0. Moreover, R(θ) > 0 for θ < θc and R(θ) < 0 for θ > θc.

Subcase 2): θ belongs to ](4r−1)π/2n, 2πr/n[. In this case, we have cos nθ ≥ 0. Then R(θ) < 0.

Consequently, if 2(2p+1)n/(4r−1) < k < (2p+1)n/(2r−1), there exists θc in ](2p+1)π/k, 2πr/n[ such that R(θc) = 0. Moreover, R(θ) > 0 for θ < θc and R(θ) < 0 for θ > θc.

Now, the three Lemmas above allow us to state the following main result for the trinomial arcs G(p, k, r, n).

**Theorem 4.6** For the trinomial arcs G(p, k, r, n), ρ(θ) is a monotonic function on the interval [(2p+1)π/k, 2πr/n].

**Proof.** Let θ be a feasible angle in ](2p+1)π/k, 2πr/n[. By Lemma 4.2, we have U(θ) < 0. On the one side, from Lemma 4.5 and Remark 4.3, we have in Case i) that R(θ) < 0. Then, the relation R(θ)[ρn − ρn−k] = U(θ)[1 − cos kθ] given by (8) implies that ρn − ρn−k > 0, which is impossible because 0 < ρ < 1. Therefore, in Case i), the hypothesis dρ/dθ = 0 is false. On the other side, Lemma 4.5 tells us that in Case ii), there exists an angle θc in ](2p+1)π/k, 2πr/n[ such that R(θc) = 0. This implies that we can distinguish the two following subcases:

Subcase 1): θ belongs to ](2p+1)π/k, θc[. In view of Lemmas 4.4 and 4.5, we have respectively that N2(θ) < 0 and R(θ) ≥ 0. So, the fact that ρ > 0 contradicts the relation R(θ) · ρn = N2(θ) of (7).

Subcase 2): θ belongs to ]θc, 2πr/n[. From Lemma 4.5, we have R(θ) < 0. Since U(θ) < 0, the equality given by (8) implies that ρn > ρn−k, which is not possible.

Accordingly, in the two cases i) and ii), the hypothesis dρ/dθ = 0 is impossible. So, we have dρ/dθ ≠ 0, i.e. ρ(θ) is monotonic on the interval [(2p+1)π/k, 2πr/n].

**Remark 4.7** When k = (2p+1)n/2r, we have (2p+1)π/k = 2πr/n. So, the trinomial arc G(p, k, r, n) is such that the bounds of the interval of feasible angles are identical. Then, this particular case corresponds to a linear trinomial arc inside the unit disk.

**Remark 4.8** When k = (2p+1)n/(2r−1), putting r′ = r − 1, we obtain that k = (2p+1)n/(2r′+1). The interval of feasible angles becomes so
Behavior of the family of trinomial arcs

According to Fell [5], we can say that this particular case corresponds to an other family of trinomial arcs solutions of (1) with $0 < \alpha < 1$, but the monotonicity of this type of arcs is never proved.

Finally, by using Theorem 4.6, we can prove the next main result.

**Theorem 4.9** $\rho(\theta)$ is an increasing function on the interval of feasible angles $[(2p+1)\pi/k, 2\pi r/n]$ for the trinomial arcs $G(p, k, r, n)$.

**Proof.** Consider a trinomial arc $G(p, k, r, n)$. If we put $\theta = (2p+1)\pi/k$ in the equation $\rho^{n-k} \sin n\theta - \rho^n \sin(n-k)\theta = \sin k\theta$ given by (4), we obtain that $\rho^{n-k}(\rho^k + 1) \sin((2p+1)\pi n/k) = 0$. Since $(2r-1)\pi < (2p+1)\pi n/k < 2\pi r$, so $\sin((2p+1)\pi n/k) \neq 0$. Then, as $\rho \geq 0$, it follows that $\rho[(2p+1)\pi/k] = 0$. On the other hand, for each trinomial arc $G(p, k, r, n)$, $\rho(\theta)$ changes between 0 and 1. Then, Theorem 4.6 allows us to deduce that $\rho(\theta)$ is an increasing function on the interval $[(2p+1)\pi/k, 2\pi r/n]$.

**References**


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