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Rectangular Group Compactification of a Semigroup

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Abstract

Among other things, a function algebra on a semitopological semigroup whose associated semigroup compactification is universal with respect to the property of being a rectangular group is studied.

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1 Introduction

In recent years there has been considerable interest in studying those function algebras on a semitopological semigroup whose associated semigroup compactifications are universal with respect to some substantial properties. For instance, see [2], [3], [5] and references therein, specially Berglund et al. [1]. Following the above mentioned papers, in this paper after some observations on the algebraic theory of simple semigroups and also rectangular groups, we introduce a function algebra $RG(S)$ on a semitopological semigroup S and as the main goal we show that RG-compactification of S is the largest rectangular compactification of S.

2 Some Algebraic Preliminaries

For the algebraic notions and primary results our basic reference is [4]; however most of them can also be found in [1]. Let X be a semigroup and let $E(X)$ be the set of all idempotents of X. An element e of $E(X)$ is said to be minimal if eXe is a subgroup of X. It is known that when X enjoys a minimal idempotent, then the minimal ideal $K(X)$ of X is non empty; moreover $E(K(X)) = \{e \in E(X) : e \text{ is minimal}\}$. A semigroup X is said to be simple if it has no proper ideal, or equivalently $K(X) = X$.
It is worth to note that, a semigroup X with a minimal idempotent e is simple if and only if X is isomorphic to the Rees product semigroup $E(Xe) \times eXe \times E(eX)$ (equipped with the Rees multiplication $(x, y, z)(u, v, w) =(x, yzuv, w)$). Indeed, the mapping $\theta(x) = (x(exe)^{-1}, exe, (exe)^{-1}x)$ is an isomorphism with the inverse $\theta^{-1}(x, y, z) = xyz$, in which, the indicated inverses are taken in the maximal subgroup eXe of X, see [1, Theorem 1.2.16] and [4, Chapter 3]. The next Proposition, which will be needed in the sequel, contains some (seemingly new) characterizations of simplicity.

Proposition 1. *For a semigroup X with a minimal idempotent, the following statements are equivalent:*

(i) X is simple.

(ii) X is weakly cancellative, that is, for $x, y, z \in X$ the equalities $xz = yz$ and $zx = zy$ imply $x = y$.

(iii) $x(exe)^{-1}x = x$, for every $x \in X$ and $e \in E(K(X))$.

(iv) There exits an element $e \in E(K(X))$ such that $x(exe)^{-1}x = x$, for all $x \in X$.

Proof. For the implication (i)⇒(ii), let X be simple and for $x, y, z \in X$, assume that $xz = yz$ and $zx = zy$. As we have remarked earlier, the mapping $\theta(x) = (x(exe)^{-1}, exe, (exe)^{-1}x)$ is an isomorphism from X onto $E(eX) \times eXe \times E(Xe)$. Thus, $(x(exe)^{-1}, exze, (eze)^{-1}z) = \theta(x)\theta(z) = \theta(xz) = \theta(yz) = \theta(y)\theta(z) = (y(eye)^{-1}, eyze, (eze)^{-1}z)$, and similarly,
$(z(eze)^{-1}, ezxe, (exe)^{-1}x) = (z(eze)^{-1}, ezye, (eye)^{-1}y)$. They follow the identities, $x(exe)^{-1} = y(eye)^{-1}$ and $(exe)^{-1}x = (eye)^{-1}y$, from which we get, $(exe)^{-1}xze = (eye)^{-1}yze$. Now the latter identity together with the identity $(exe)((exe)^{-1}xze) = exze = eyze = (eye)((eye)^{-1}yze)$ lead us the equality $exe = eye$. Thus $\theta(x) = \theta(y)$, or equivalently $x = y$.

For (ii)⇒(iii), since for every $x \in X$ and $e \in E(K(X))$ we have $x(exe)^{-1}xe = xe$ and $ex(exe)^{-1}x = ex$, the equality $x(exe)^{-1}x = x$ follows from the weak cancelation property of X. (iii)⇒(iv) is obvious. For (iv)⇒(i), note that eXe lies in $K(X)$. Therefore for all $x \in X$, $x = x(exe)^{-1}x \in K(X)$; in other words, X is simple.

Remark 2. From the proof of the implication (i)⇒(ii) (in Proposition 1), it can be extracted that, if X is simple then the equality $xz = yz$ for some $z \in X$ implies that $(exe)^{-1}x = (eye)^{-1}y$. We will use this fact in the proof of Proposition 7.

A semigroup X is said to be a rectangular group if it is isomorphic to the direct product of a left zero semigroup, a group and a right zero semigroup.

Let Y be a subsemigroup of X, then X is said to be an inflation of Y if $X^2 \subseteq Y$ and there exists an epimorphism $r : X \to Y$ whose restriction to Y is the identity mapping.

Proposition 3. *Let X be a semigroup, then*

(i) X is a rectangular group if and only if X has a minimal idempotent, X is simple and $xey = xy$, for all $x, y \in X$, $e \in E(X)$.

(ii) X is an inflation of a rectangular group if and only if X has a minimal idempotent and $xey = xy$, for all $x, y \in X$, $e \in E(X)$.

Proof. (i) Let $X = L \times G \times R$ be a rectangular group, where L is a left zero semigroup, R is a right zero semigroup and G is a group. Now, a direct verification reveals that, $xey = xy$ (for all $x, y \in X$ and $e \in E(X)$) and also $E(X) = L \times \{1\} \times R$, where 1 is the identity of G. Moreover, since $\{L \times G \times \{r\}; r \in R\}$ is the set of all minimal left ideals of X, every element of $E(X)$ is minimal and also, $K(X) = \cup\{L \times G \times \{r\}; r \in R\} = X$; that is, X is simple. For the converse, since X is simple and it contains a minimal idempotent, X is isomorphic to the Rees product semigroup $E(Xe) \times eXe \times E(eX)$. But, the identity $xey = xy$ in X implies that the Rees product and the direct product coincide on $E(Xe) \times eXe \times E(eX)$. In particular, X is a rectangular group.

(ii) Let X be an inflation of a rectangular group Y, and let $r : X \to Y$ be the involved epimorphism. Since $X^2 \subseteq Y$, one can verify that every minimal idempotent of Y is also minimal in X. Moreover, for all $x, y \in X$ and $e \in E(X)$, we have $xey \in Y$ and so $xey = r(xey) = r(x)r(e)r(y) = r(x)r(y) = r(xy) = xy$. For the converse, let e be in $E(K(X))$, then $K(X)$ is isomorphic to the Rees product semigroup $E(Xe) \times eXe \times E(eX)$, see [1, Theorem 1.2.16]. The identity $xey = xy$ in X implies that the Rees product and the direct product on $E(Xe) \times eXe \times E(eX)$ coincide. In particular, $K(X)$ is a rectangular group. Trivially $X^2 = XeX = K(X)$. Now the mapping $r(x) = x(exe)^{-1}x : X \to K(X)$ is onto, its restriction to $K(X)$ is identity and also $r(x)r(y) = x(exe)^{-1}xy(eye)^{-1}y = x(exe)^{-1}exeeye(eye)^{-1}y = xey = xy = r(xy)$. Therefore, X is an inflation of the rectangular group $K(X)$.

Corollary 4. *A semigroup X is a rectangular group if and only if X is* *both simple and an inflation of a rectangular group.*

Proof. Follows trivially from the last Proposition.

3 The Main Results

In what follows, S will be at least a semitopological semigroup. For $C(S)$ (= the C^*-algebra of all bounded continuous complex valued functions on S) the left and right translations L_s and R_t are defined for all $s, t \in S$ and $f \in C(S)$ by $(L_s f)(t) = f(st) = (R_t f)(s)$. A unital C^*-subalgebra F of $C(S)$ is said to be m-admissible if it is both left translation invariant (that is, $L_s f \in F$ for all $s \in S$ and $f \in F$) and left m-introverted (that is, $T_\mu f \in F$, for all $f \in F$ and $\mu \in S^F =$ the spectrum of F). If so, then the pair (ϵ, S^F) is a semigroup compactification of S, in which $\epsilon : S \to S^F$ is the evaluation mapping and S^F is endowed with the $weak^*$ topology and the multiplication $\mu\nu = \mu o T_\nu$. The m-admissible algebras of distal, strongly distal and minimal distal functions on S are denoted by $D(S), SD(S)$ and $MD(S)$, respectively (but we usually suppress the letter S, whenever there is no risk of ambiguity). We write GP for $SD \cap MD$. The associated compactifications of these function algebras are universal with respect to the property of being an inflation of a rectangular group, right simple semigroup, left simple semigroup and a group, respectively. For more details see [1, Sec. 4.6] and [5, Theorem 3.4].

Now we come to the main aim of the present paper which introduce a function algebra $RG(S)$ and examines it from the universal semigroup compactification point of view.

Definition 5. We define $RG(S)$ as the set of all f in $D(S)$ such that, for every two given nets $\{t_\alpha\}$ and $\{u_\alpha\}$ in S, the equalities:

$\lim_\alpha(\lim_\alpha R_{s_\alpha} f)(t_\alpha) = \lim_\alpha(\lim_\alpha R_{s_\alpha} f)(u_\alpha)$, and
$\lim_\alpha(\lim_\alpha R_{t_\alpha} f)(s_\alpha) = \lim_\alpha(\lim_\alpha R_{u_\alpha} f)(s_\alpha)$,
for all nets $\{s_\alpha\}$ in S necessitates $\lim_\alpha f(t_\alpha) = \lim_\alpha f(u_\alpha)$ (the used limits on the right translations are assumed to be pointwise).

We will use the following Lemma in the proof of the Proposition 7.

Lemma 6. *Let $f \in D$ be such that $\eta(e\eta e)^{-1}\eta(f) = \eta(f)$, for all $\eta \in S^D$, and let F be the m-admissible subalgebra of $C(S)$ generated by $\{T_\eta f : \eta \in S^D\}$. Then S^F is simple.*

Proof. Trivially $F \subseteq D$. Let $\mu \in S^F$ and $e \in E(S^F)$ be minimal. Then for every $\eta \in S^D$, $\mu(e\mu e)^{-1}\mu(T_\eta f) = \mu(e\mu e)^{-1}\mu\eta(f) = \mu(e\mu e)^{-1}e\mu e\eta(f) = \mu e\eta(f) = \mu\eta(f) = \mu(T_\eta f)$. Therefore, $\mu(e\mu e)^{-1}\mu(g) = \mu(g)$, for all $g \in F$. Now part (iii) of Proposition 1 implies that S^F is simple.

The next result describes RG in terms of the elements of S^D. The inverses in parts (iii) and (iv) are taken in the group eS^De.

Proposition 7. *For every $f \in D$, the following statements are equivalent:*

(i) $f \in RG$.

(ii) For a given $\mu, \nu \in S^D$, the equalities $\mu\eta(f) = \nu\eta(f)$, and $\eta\mu(f) = \eta\nu(f)$, for all $\eta \in S^D$ imply $\mu(f) = \nu(f)$.

(iii) For every $e \in E(K(S^D))$, and all $\mu \in S^D$, $\mu(e\mu e)^{-1}\mu(f) = \mu(f)$.

(iv) There exists an $e \in E(K(S^D))$ such that for all $\mu \in S^D$, $\mu(e\mu e)^{-1}\mu(f) = \mu(f)$.

Proof. That (i) is equivalent to (ii) is straightforward. For $(ii) \Rightarrow (iii)$ assume that $\mu \in S^D$ and $e \in E(K(S^D))$. Then for all $\eta \in S^D$, we have $(\mu(e\mu e)^{-1}\mu)\eta(f) = \mu(e\mu e)^{-1}e\mu e\eta(f) = \mu e\eta(f) = \mu\eta(f)$ and $\eta(\mu(e\mu e)^{-1}\mu)(f) = \eta e\mu e(e\mu e)^{-1}\mu(f) = \eta\mu(f)$. Thus the identity $\mu(e\mu e)^{-1}\mu(f) = \mu(f)$ follows from (ii).

The implication $(iii) \Rightarrow (iv)$ is trivial. For $(iv) \Rightarrow (ii)$, assume that $\mu, \nu \in S^D$ such that $\mu\eta(f) = \nu\eta(f)$, and $\eta\mu(f) = \eta\nu(f)$, for all $\eta \in S^D$. Suppose that F is the m-admissible subalgebra of $C(S)$ generated by $\{T_\eta f : \eta \in S^D\}$; then $F \subseteq D$ and (as Lemma 6 demonstrates), the semigroup S^F is simple. By what we have mentioned in Remark 2, the equality $\mu\eta = \nu\eta$ in S^F implies $\mu(e\mu e)^{-1} = \nu(e\nu e)^{-1}$ in S^F. In particular, we have $\mu(e\mu e)^{-1}\mu(f) = \nu(e\nu e)^{-1}\mu(f)$. From this and the fact that $\eta\mu(f) = \eta\nu(f)$, for all $\eta \in S^D$, we get $\mu(f) = \mu(e\mu e)^{-1}\mu(f) = \nu(e\nu e)^{-1}\mu(f) = \nu(e\nu e)^{-1}\nu(f) = \nu(f)$, as claimed.

The next theorem which is the main result of the present paper examines RG from the semigroup compactification point of view.

Theorem 8. *$RG(S)$ is an m-admissible subalgebra of $C(S)$, and (ϵ, S^{RG}) is the universal rectangular group compactification of S.*

Proof. For $f \in RG(S)$, $\mu, \nu, \eta \in S^D$, and $e \in E(K(S^D))$ we have trivially the identities, $\eta\mu(e\mu e)^{-1}\mu(f)=\eta e\mu e(e\mu e)^{-1}\mu(f)=\eta e\mu(f)=\eta\mu(f)$, and $\mu(e\mu e)^{-1}\mu\eta(f)=\mu(e\mu e)^{-1}e\mu e\eta(f)=\mu e\eta(f)=\mu\eta(f)$, which necessity RG is left m-introverted. Part (iii) of Proposition 7 implies that $RG(S)$ is a unital C^*-subalgebra of $D(S)$, and for all $\mu, \nu \in S^{RG}$, and $e \in E(K(S^{RG}))$, $\mu e\nu = \mu\nu$ and $\mu(e\mu e)^{-1}\mu = \mu$. Now part (i) of Proposition 3 together with part (iii) of Proposition 1 necessity S^{RG} is a rectangular group. To prove the universal property of (ϵ, S^{RG}) it remains to show that for every other rectangular group compactification (ψ, X) of S, $\psi^*(C(X)) \subseteq RG(S)$. For this end, since (ψ, X) is a distal compactification $\psi^*(C(X)) \subseteq D(S)$. Let $\pi : S^D \to X$ be the canonical homomorphism. Then for every $f \in C(X)$, $\mu \in S^D$ and $e \in E(K(S^D))$ we have $\mu(e\mu e)^{-1}\mu(\psi^*(f))$=$f(\pi(\mu(e\mu e)^{-1}\mu)) = f(\pi(\mu)(\pi(e)\pi(\mu)\pi(e))^{-1}\pi(\mu))$= $f(\pi(\mu)) = \mu(\psi^*(f))$. Therefore $\psi^*(f) \in RG$, as claimed.

Corollary 9. *If S is a compact rectangular group then $RG(S) = C(S)$.*

Proof. Since S is a compact semitopological semigroup, the mapping $\epsilon : S \to S^{C(S)}$ is surjective, and so $S^{C(S)}$ is a rectangular group when S is a rectangular group. Now the universal property of S^{RG} (Theorem 8) implies that $RG(S) = C(S)$.

Remarks and examples 10. (i) Mimic the methods of Proposition 3.10 of [5], one can verify that $RG = \langle LZ \cup GP \cup RZ \rangle = LZ \otimes GP \otimes RZ$, where $LZ = \{f \in C(S), f(st) = f(s); \forall s, t \in S\}$ and $RZ = \{f \in C(S), f(st) = f(t); \forall s, t \in S\}$, $\langle LZ \cup GP \cup RZ \rangle$ is the C^*-subalgebra of $C(S)$ generated by $LZ \cup GP \cup RZ$ and $\otimes$ is used for the topological tensor product.

(ii) The universal properties of MD and SD-compactifications together with Theorem 8 imply that $MD \subseteq RG$ and $SD \subseteq RG$, and so $GP \subseteq RG$.

(iii) Using Part (ii) of Proposition 7, one can show that for a null semigroup S, $D(S) = C(S)$ and $GP(S) = RG(S) = \mathbf{C}$ (= the constant functions). Also, if S is a left zero (or a right zero) semigroup then $RG(S) = D(S) = C(S)$ and $GP(S) = \mathbf{C}$.

(iv) It would be desirable to construct a function algebra, say SP, (probably, with the same techniques) whose associated compactification is the universal simple compactification. Of course SP will satisfy in the equality $D \cap SP = RG$, by Part (i) of Proposition 3.

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