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A Generalized Integration by Parts[1]

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Abstract

In this article a generalization of integration by parts for the Riemann-Stieltjes integral is presented.

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Introduction

Fundamental Theorem of Calculus. Suppose F is a real-valued function which is differentiable on [a,b]. If the derivative F' is Riemann integrable over [a,b], then $\int_a^b F'(x)dx = F(b) - F(a)$.

This theorem appears in every classical analysis book and is extended in some directions. For instance, when F is right differentiable on $[a, b)\backslash D$, where D is a countable subset of [a,b], then under some extra conditions the theorem is valid. We may, also have the theorem whenever F is a complex-valued function that is absolutely continuous on [a,b] see for example [5, P.311, 320]. See also [1]-[4], [6] and [7]. In this paper we bring some conditions under which the fundamental theorem of calculus and integration by parts are valid for the Riemann-Stieltjes integral.

Main Results

Definition 1. Let the real valued function f be defined on [a,b]. Also, let φ be an strictly increasing function on [a,b]. For any $x \in [a, b]$ we define

$$D(f, \varphi)(x) = \lim_{t \to x} \frac{f(t) - f(x)}{\varphi(t) - \varphi(x)}.$$

Moreover, we denote the Riemann-Stieltjes integral of f with respect to φ over [a,b] by $\int_a^b f d\varphi$. If $\varphi(x) = x$ then $\int_a^b f d\varphi = \int_a^b f dx$ is the Riemann integral of f over [a,b]. It can be easily seen that for functions f and g the following properties hold:

Property 1. $D(fg, \varphi) = D(f, \varphi)g + fD(g, \varphi)$.

Property 2. $D(f/g, \varphi) = \frac{D(f,\varphi)g - D(g,\varphi)f}{g^2}$ $\quad g(x) \neq 0 \ \ \forall x \in [a, b]$.

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Property 3. Suppose that f is continuous on [a,b]; and φ is an strictly in creasing continuous function defined on [a,b]. Furthermore, let g be defined on an interval I which contains the range of f and $g'(f(x))$ exists on [a,b]. If $h(t) = g(f(t))$ then $D(h, \varphi)(x)$ exists and

$$D(h, \varphi)(x) = g'(f(x))D(f, \varphi)(x) \quad \text{for all} \quad x \in [a, b].$$

Theorem 1. Let φ be an strictly increasing continuous function on [a,b]. Suppose that f is a Riemann-Stieltjes integrable function with respect to φ on [a,b]. If there is a function F on [a,b] such that

$$D(F, \varphi)(x) = f(x), \forall x \in [a, b]$$

then

$$\int_a^b f d\varphi = F(b) - F(a).$$

Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ be a partition of [a,b]. Fix the interval $[x_{i-1}, x_i]$ and put

$$g(t) = (F(x_i) - F(x_{i-1})\varphi(t) - (x_i - x_{i-1})F(t),\ x_{i-1} \leq t \leq x_i.$$

Continuity of φ implies that F is continuous on $[x_{i-1}, x_i]$, and so is g. Also,

$$D(g, \varphi)(t) = (F(x_i) - F(x_{i-1})) - (x_i - x_{i-1})f(t), \quad x_{i-1} \leq t \leq x_i,$$

and $g(x_i) = g(x_{i-1})$. Suppose $g(t) > g(x_{i-1})$ for some $t \in (x_{i-1}, x_i)$. Let x be a point on $[x_{i-1}, x_i]$ at which g attains its maximum. So $x \in (x_{i-1}, x_i)$ and

$$\frac{g(t) - g(x)}{\varphi(t) - \varphi(x)} \geq 0 \qquad x_{i-1} < t < x.$$

Letting $t \to x$, we have $D(g, \varphi) \geq 0$. On the other hand,

$$\frac{g(t) - g(x)}{\varphi(t) - \varphi(x)} \leq 0 \quad (x < t < x_i)$$

which shows that $D(g, \varphi)(x) \leq 0$. Thus, $D(g, \varphi)(x) = 0$. Similarly, if $g(t) < g(x_{i-1})$ for some $t \in (x_{i-1}, x_i)$, assume that x is a point in $[x_{i-1}, x_i]$ at which g attains its minimum. So $x \in (x_{i-1}, x_i)$ and $D(g, \varphi)(x) = 0$. Furthermore if g is a constant function on $[x_{i-1}, x_i]$ then $D(g, \varphi)(x) = 0$ for all x in (a,b). Thus, we conclude that there is a point $t_i \in (x_{i-1}, x_i)$ so that

$$F(x_i) - F(x_{i-1}) = (\varphi(x_i) - \varphi(x_{i-1}))f(t_i).$$

Fix $\varepsilon > 0$. Since f is Riemann-Stieltjes integrable with respect to φ on [a,b] we have

$$|\sum_{i=1}^{n}(\varphi(x_i) - \varphi(x_{i-1}))f(t_i) - \int_a^b f d\varphi| < \varepsilon,$$

for some partition $\{x_o = a < x_1 < \cdots < x_{n-1} < x_n = b\}$. Hence

$$\int_a^b f d\varphi = \sum_{i=1}^n F(x_i) - F(x_{i-1}) = F(b) - F(a).\Box$$

Corollary 1. If φ is an strictly increasing continuous function on [a,b], and $\varphi(x) \neq 0 \ \forall x \in [a, b]$ then

$$\int_a^b \frac{d\varphi}{\varphi} = (ln|\varphi|)(b) - (ln|\varphi|)(a).$$

Proof. Let $f(x) = |\varphi(x)|$ and $g(x) = lnx$ in Property 3. Then $D(h, \varphi)(x) = \frac{1}{|\varphi(x)|}D(|\varphi|, \varphi)(x) = \frac{1}{\varphi(x)}$ where $h(x) = ln|\varphi(x)|$. Now, apply Theorem 1.

Corollary 2. If φ is an strictly increasing continuous function on [a,b] such that $\varphi(x) \neq 0, \forall x \in [a, b]$, then

$$\int_a^b \varphi^n d\varphi = \frac{\varphi^{n+1}(b)}{n+1} - \frac{\varphi^{n+1}(a)}{n+1}$$

for any real number $n \neq 1$.
Proof. By the proof of Corollary 1 we have

$$D(ln|\varphi^n|, \varphi) = nD(ln|\varphi|, \varphi) = \frac{n}{\varphi}.$$

Now, using Property 3 for $f(x) = \varphi(x)^n$ and $g(x) = ln|x|$ we have $D(ln|\varphi^n|, \varphi) = \frac{1}{\varphi^n}D(\varphi^n, \varphi)$, so $D(\varphi^n, \varphi) = n\varphi^{n-1}$. Therefore, $D(\frac{\varphi^{n+1}}{n+1}, \varphi) = \varphi^n, n \neq -1$. Thus the result holds by Theorem 1.

Theorem 2(Generalized Integration by Parts). Let φ be an strictly increasing continuous function on [a,b]. Suppose that F and G are functions defined on [a,b] such that $D(F, \varphi)(x)$ and $D(G, \varphi)(x)$ exist for every $a \leq x \leq b$, and are Riemann-Stieltjes integrable with respect to φ on [a,b]. Then

$$\int_a^b FD(G, \varphi)d\varphi = F(b)G(b) - F(a)G(a) - \int_a^b GD(F, \varphi)d\varphi.$$

Proof. Since φ is continuous and $D(F, \varphi)$ and $D(G, \varphi)$ exist, we conclude that F and G are continuous on [a,b]. Therefore, $FD(G, \varphi)$ and $GD(F, \varphi)$ are Riemann-Stieltjes integrable with respect to φ. Put $H(x) = F(x)G(x)$. By Property 1 we have

$$D(H, \varphi) = FD(G, \varphi) + GD(F, \varphi)$$

and Theorem 1 implies that

$$\int_a^b D(H, \varphi)d\varphi = H(b) - H(a).$$

So the result holds. $\square$

Theorem 3. Let φ be a strictly increasing function on [a,b] and suppose that f is Riemann-Stieltjes integrable with respect to φ on [a,b]. For $a \leq x \leq b$, put $F(x) = \int_a^x f d\varphi$. Then

(a) If f is continuous at x_0 of [a,b], then

$$D(F,\varphi)(x_0) = f(x_0)$$

(b) If φ is continuous on [a,b], then F is continuous on [a,b].

Proof. (a) Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon$$

if $|t - x_0| < \delta$, and $a \leq t \leq b$. Hence, if

$$x_0 - \delta < s \leq x_0 \leq t < x_0 + \delta \quad \text{and}$$

$a \leq s < t \leq b$ we have

$$|\frac{F(t) - F(s)}{\varphi(t) - \varphi(s)} - f(x_0)| = |\frac{1}{\varphi(t) - \varphi(s)} \int_s^t [f(u) - f(x)] d\varphi| < \varepsilon$$

It follows that $D(F,\varphi)(x_0) = f(x_0)$.

(b) Since f is bounded suppose that

$$|f(t)| \leq M \quad \text{for} \quad a \leq t \leq b. \text{ If } \quad a \leq x < y \leq b,$$

then $|F(y) - F(x)| = |\int_x^y f d\varphi| \leq M |\int_x^y d\varphi| = M|\varphi(y) - \varphi(x)|$. Since φ is uniformly continuous for every $\varepsilon > 0$ we see that $|F(y) - F(x)| < \varepsilon$, when $|x - y| < \delta$ for some $\delta > 0$. This proves uniformly continuity of F on [a,b].$\square$

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