A Generalized Integration by Parts

K. Hedayatian

Department of Mathematics
Shiraz University, Shiraz 71454, Iran
hedayati@shirazu.ac.ir

Abstract

In this article a generalization of integration by parts for the Riemann-Stieltjes integral is presented.

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Introduction

Fundamental Theorem of Calculus. Suppose $F$ is a real-valued function which is differentiable on $[a,b]$. If the derivative $F'$ is Riemann integrable over $[a,b]$, then

$$\int_a^b F'(x)dx = F(b) - F(a).$$

This theorem appears in every classical analysis book and is extended in some directions. For instance, when $F$ is right differentiable on $[a, b) \setminus D$, where $D$ is a countable subset of $[a,b]$, then under some extra conditions the theorem is valid. We may also have the theorem whenever $F$ is a complex-valued function that is absolutely continuous on $[a,b]$ see for example [5, P.311, 320]. See also [1]-[4], [6] and [7]. In this paper we bring some conditions under which the fundamental theorem of calculus and integration by parts are valid for the Riemann-Stieltjes integral.

Main Results

Definition 1. Let the real valued function $f$ be defined on $[a,b]$. Also, let $\varphi$ be an strictly increasing function on $[a,b]$. For any $x \in [a,b]$ we define

$$D(f, \varphi)(x) = \lim_{t \to x} \frac{f(t) - f(x)}{\varphi(t) - \varphi(x)}.$$

Moreover, we denote the Riemann-Stieltjes integral of $f$ with respect to $\varphi$ over $[a,b]$ by $\int_a^b f d\varphi$. If $\varphi(x) = x$ then $\int_a^b f d\varphi = \int_a^b f dx$ is the Riemann integral of $f$ over $[a,b]$. It can be easily seen that for functions $f$ and $g$ the following properties hold:

Property 1. $D(fg, \varphi) = D(f, \varphi)g + fD(g, \varphi)$.

Property 2. $D(f/g, \varphi) = \frac{D(f, \varphi)g - fD(g, \varphi)}{g} \quad g(x) \neq 0 \quad \forall x \in [a, b]$. 

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Property 3. Suppose that \( f \) is continuous on \([a,b]\); and \( \varphi \) is an strictly increasing continuous function defined on \([a,b]\). Furthermore, let \( g \) be defined on an interval \( I \) which contains the range of \( f \) and \( g'(f(x)) \) exists on \([a,b]\). If \( h(t) = g(f(t)) \) then \( D(h, \varphi)(x) \) exists and
\[
D(h, \varphi)(x) = g'(f(x))D(f, \varphi)(x) \quad \text{for all} \quad x \in [a,b].
\]

**Theorem 1.** Let \( \varphi \) be an strictly increasing continuous function on \([a,b]\). Suppose that \( f \) is a Riemann-Stieltjes integrable function with respect to \( \varphi \) on \([a,b]\). If there is a function \( F \) on \([a,b]\) such that
\[
D(F, \varphi)(x) = f(x), \forall x \in [a,b]
\]
then
\[
\int_a^b f \, d\varphi = F(b) - F(a).
\]
**Proof.** Let \( P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\} \) be a partition of \([a,b]\). Fix the interval \([x_{i-1}, x_i]\) and put
\[
g(t) = (F(x_i) - F(x_{i-1}))\varphi(t) - (x_i - x_{i-1})F(t), \quad x_{i-1} \leq t \leq x_i.
\]
Continuity of \( \varphi \) implies that \( F \) is continuous on \([x_{i-1}, x_i]\), and so is \( g \). Also,
\[
D(g, \varphi)(t) = (F(x_i) - F(x_{i-1})) - (x_i - x_{i-1})f(t), \quad x_{i-1} \leq t \leq x_i,
\]
and \( g(x_i) = g(x_{i-1}) \). Suppose \( g(t) > g(x_{i-1}) \) for some \( t \in (x_{i-1}, x_i) \). Let \( x \) be a point on \([x_{i-1}, x_i]\) at which \( g \) attains its maximum. So \( x \in (x_{i-1}, x_i) \) and
\[
g(t) - g(x) \\[\varphi(t) - \varphi(x)] \geq 0 \quad x_{i-1} < t < x.
\]
Letting \( t \rightarrow x \), we have \( D(g, \varphi) \geq 0 \). On the other hand,
\[
\frac{g(t) - g(x)}{\varphi(t) - \varphi(x)} \leq 0 \quad (x < t < x_i)
\]
which shows that \( D(g, \varphi)(x) \leq 0 \). Thus, \( D(g, \varphi)(x) \geq 0 \). Similarly, if \( g(t) < g(x_{i-1}) \) for some \( t \in (x_{i-1}, x_i) \), assume that \( x \) is a point in \([x_{i-1}, x_i]\) at which \( g \) attains its minimum. So \( x \in (x_{i-1}, x_i) \) and \( D(g, \varphi)(x) = 0 \). Furthermore if \( g \) is a constant function on \([x_{i-1}, x_i]\) then \( D(g, \varphi)(x) = 0 \) for all \( x \) in \((a,b)\). Thus, we conclude that there is a point \( t_i \in (x_{i-1}, x_i) \) so that
\[
F(x_i) - F(x_{i-1}) = (\varphi(x_i) - \varphi(x_{i-1}))f(t_i).
\]
Fix \( \varepsilon > 0 \). Since \( f \) is Riemann-Stieltjes integrable with respect to \( \varphi \) on \([a,b]\) we have
\[
\left| \sum_{i=1}^n (\varphi(x_i) - \varphi(x_{i-1}))f(t_i) - \int_a^b f \, d\varphi \right| < \varepsilon,
\]
for some partition \( \{ x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b \} \). Hence

\[
\int_a^b f d\varphi = \sum_{i=1}^n F(x_i) - F(x_{i-1}) = F(b) - F(a). \quad \square
\]

**Corollary 1.** If \( \varphi \) is an strictly increasing continuous function on \([a,b]\), and \( \varphi(x) \neq 0 \ \forall x \in [a,b] \) then

\[
\int_a^b \frac{d\varphi}{\varphi} = (\ln|\varphi|)(b) - (\ln|\varphi|)(a).
\]

**Proof.** Let \( f(x) = |\varphi(x)| \) and \( g(x) = \ln x \) in Property 3. Then \( D(h, \varphi)(x) = \frac{1}{|\varphi(x)|} D(|\varphi|, \varphi)(x) = h(x) = \ln|\varphi(x)| \). Now, apply Theorem 1.

**Corollary 2.** If \( \varphi \) is an strictly increasing continuous function on \([a,b]\) such that \( \varphi(x) \neq 0, \forall x \in [a,b] \), then

\[
\int_a^b \varphi^n d\varphi = \frac{\varphi^{n+1}(b)}{n+1} - \frac{\varphi^{n+1}(a)}{n+1}
\]

for any real number \( n \neq 1 \).

**Proof.** By the proof of Corollary 1 we have

\[
D(ln|\varphi^n|, \varphi) = nD(ln|\varphi|, \varphi) = \frac{n}{\varphi}.
\]

Now, using Property 3 for \( f(x) = \varphi(x)^n \) and \( g(x) = \ln|x| \) we have \( D(ln|\varphi^n|, \varphi) = \frac{1}{\varphi} D(\varphi^n, \varphi), \) so \( D(\varphi^n, \varphi) = n\varphi^{n-1} \). Therefore, \( D(\frac{\varphi^{n+1}}{n+1}, \varphi) = \varphi^n, n \neq -1 \). Thus the result holds by Theorem 1.

**Theorem 2 (Generalized Integration by Parts).** Let \( \varphi \) be an strictly increasing continuous function on \([a,b]\). Suppose that \( F \) and \( G \) are functions defined on \([a,b]\) such that \( D(F, \varphi)(x) \) and \( D(G, \varphi)(x) \) exist for every \( a \leq x \leq b \), and are Riemann-Stieltjes integrable with respect to \( \varphi \) on \([a,b]\). Then

\[
\int_a^b F(D(G, \varphi)) d\varphi = F(b)G(b) - F(a)G(a) - \int_a^b G(D(F, \varphi)) d\varphi.
\]

**Proof.** Since \( \varphi \) is continuous and \( D(F, \varphi) \) and \( D(G, \varphi) \) exist, we conclude that \( F \) and \( G \) are continuous on \([a,b]\). Therefore, \( FD(G, \varphi) \) and \( GD(F, \varphi) \) are Riemann-Stieltjes integrable with respect to \( \varphi \). Put \( H(x) = F(x)G(x) \). By Property 1 we have

\[
D(H, \varphi) = FD(G, \varphi) + GD(F, \varphi)
\]

and Theorem 1 implies that

\[
\int_a^b D(H, \varphi) d\varphi = H(b) - H(a).
\]
So the result holds. □

**Theorem 3.** Let \( \varphi \) be a strictly increasing function on \([a,b]\) and suppose that \( f \) is Riemann-Stieltjes integrable with respect to \( \varphi \) on \([a,b]\). For \( a \leq x \leq b \), put \( F(x) = \int_a^x f \, d\varphi \). Then

(a) If \( f \) is continuous at \( x_0 \) of \([a,b]\), then

\[
D(F, \varphi)(x_0) = f(x_0)
\]

(b) If \( \varphi \) is continuous on \([a,b]\), then \( F \) is continuous on \([a,b]\).

**Proof.** (a) Given \( \varepsilon > 0 \), choose \( \delta > 0 \) such that

\[
|f(t) - f(x_0)| < \varepsilon
\]

if \( |t - x_0| < \delta \), and \( a \leq t \leq b \). Hence, if

\[
x_0 - \delta < s \leq x_0 \leq t < x_0 + \delta
\]
and

\[
a \leq s < t \leq b
\]

we have

\[
\left| \frac{F(t) - F(s)}{\varphi(t) - \varphi(s)} - f(x_0) \right| = \left| \frac{1}{\varphi(t) - \varphi(s)} \int_s^t [f(u) - f(x)] \, d\varphi \right| < \varepsilon
\]

It follows that \( D(F, \varphi)(x_0) = f(x_0) \).

(b) Since \( f \) is bounded suppose that

\[
|f(t)| \leq M \quad \text{for} \quad a \leq t \leq b.
\]
if \( a \leq x < y \leq b \), then

\[
|F(y) - F(x)| = \left| \int_x^y f \, d\varphi \right| \leq M \left| \int_x^y d\varphi \right| = M|\varphi(y) - \varphi(x)|.
\]

Since \( \varphi \) is uniformly continuous for every \( \varepsilon > 0 \) we see that \( |F(y) - F(x)| < \varepsilon \), when \( |x - y| < \delta \) for some \( \delta > 0 \). This proves uniformly continuity of \( F \) on \([a,b]\).

REFERENCES


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