Evaluation of Certain Definite Integrals involving Trigonometric Functions

K. Hedayatian

Department of Mathematics
College of Sciences
Shiraz University, Shiraz 71454, Iran
hedayati@shirazu.ac.ir

M. Faghih Ahmadi

Islamic Azad University-Sepidan Branch, Iran

Abstract

This paper is concerned with integrating of two classes of trigonometric functions using line integration. The first class, also gives us a formula involving arc tangent functions.

Mathematics Subject Classification: 33B10

Keywords: Green’s theorem, Definite trigonometric integrals

Introduction

Besides its own interest, integration of trigonometric functions has important applications in many branches of science. Except the usual ways, there are many other techniques to evaluate definite integrals. For instance, one way is to apply residue theorem [3], [4]. Using a recursion formula is another way [8]. It expresses the integral of a power of a function in terms of the integral of a lower power of the function. J. P. McCammond in 1999, gave a method for integrating polynomials in tangent and secant [7]. Some other techniques can be studied in [1], [2], and [6].

In this article, Green’s theorem, one of the great, surprising theorems of calculus, is used to compute the integral of some rational functions of sine and...
Green’s Theorem. (Circulation, curl, or tangential form) The counterclockwise circulation of $F = M\mathbf{i} + N\mathbf{j}$ around a simple closed curve $C$ in the direction of its unit tangent vector $\mathbf{T}$, in the plane is the double integral of $(\text{curl}\,F)\,\mathbf{k}$ over the region $R$ enclosed by $C$;

$$\oint_C F \cdot \mathbf{T} \, ds = \oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.$$ 

Main Results

**Theorem 1.** If $a$ is a positive real number and $0 < \alpha \leq \beta < \pi / 2$, then

$$\int_\alpha^\beta \frac{\sin^{a-1}\theta \cos^{a-1}\theta}{\cos^a \theta + \sin^a \theta} \, d\theta = \left[ \frac{\pi}{2} - \tan^{-1}\left( \frac{x_\alpha + m^2 x_\alpha - m^2}{m} \right) + \tan^{-1}\left( \frac{1}{m} \right) \right. - \tan^{-1}(x_\beta + x_\beta n^2 + n) + \tan^{-1}n \right] \frac{a}{2},$$

where

$$x_\alpha = \left( \frac{1}{1 + (\tan\alpha)^a} \right)^{\frac{1}{2}}, \quad x_\beta = \left( \frac{1}{1 + (\tan\beta)^a} \right)^{\frac{1}{2}},$$

$$m = \frac{x_\alpha \tan\alpha}{x_\alpha - 1}, \quad n = \frac{x_\beta \tan\beta - 1}{x_\beta}.$$ 

**Proof.** Let $C = \bigcup_{i=1}^{7} C_i$ be a positively oriented curve where $C_i, (i = 1, 2, 3)$ are the line segments from the point $(0, 1)$ to $(-1, 0)$ and $(-1, 0)$ to $(0, -1)$ and $(0, -1)$ to $(1, 0)$, respectively. By taking $y_\alpha = x_\alpha \tan \alpha$ and $y_\beta = x_\beta \tan \beta$, let $C_4$ be the line segment from the point $(1, 0)$ to $(x_\alpha, y_\alpha)$ and $C_5$ be a piece of the curve $x^a + y^a = 1$ from the point $(x_\alpha, y_\alpha)$ to the point $(x_\beta, y_\beta)$. Also $C_6$ is the line segment from the point $(x_\beta, y_\beta)$ to $(0, 1)$ and $C_7$ is the circle $x^2 + y^2 = r^2$ where $r > 0$ is sufficiently small so that $C_7$ lies inside the closed curve $\bigcup_{i=1}^{6} C_i$. Putting

$$M(x, y) = \frac{-y}{x^2 + y^2}, \quad N(x, y) = \frac{x}{x^2 + y^2}$$

and then applying Green’s theorem, we conclude that

$$\oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = 0,$$

where $R$ is the region enclosed by $C$. Thus

$$\sum_{i=1}^{7} \oint_{C_i} M \, dx + N \, dy = 0.$$
But
\[
\int_{C_1} Mdx + Ndy = \int_0^{-1} \frac{-1-x}{x^2 + (1+x)^2} dx + \int_0^{-1} \frac{x}{x^2 + (1+x)^2} dx
\]
\[= \int_0^{-1} \frac{-1}{x^2 + (1+x)^2} dx\]

and
\[
\int_{C_2} Mdx + Ndy = \int_{-1}^0 \frac{x+1}{x^2 + (x+1)^2} dx - \int_{-1}^0 \frac{x}{x^2 + (1+x)^2} dx
\]
\[= \int_{-1}^0 \frac{dx}{(1+x)^2}.
\]

and
\[
\int_{C_3} Mdx + Ndy = \int_{0}^{1} \frac{1-x}{x^2 + (x-1)^2} dx + \int_{0}^{1} \frac{x}{x^2 + (x-1)^2} dx
\]
\[= \int_{0}^{1} \frac{dx}{x^2 + (x-1)^2}.
\]

Furthermore,
\[
\int_{0}^{-1} \frac{dx}{x^2 + (1+x)^2} = \frac{1}{2} \int_{0}^{-1} \frac{dx}{(x+\frac{1}{2})^2 + \frac{1}{4}} = \tan^{-1}(-1) - \tan^{-1}(1) = -\frac{\pi}{2}.
\]

Therefore,
\[
\sum_{i=1}^{3} \int_{C_i} Mdx + Ndy = \pi + \int_{0}^{1} \frac{dx}{x^2 + (x-1)^2} = \pi - \int_{0}^{-1} \frac{dx}{x^2 + (x+1)^2} = \frac{3\pi}{2}.
\]

On the other hand, since \(C_4\) is defined by \(y = mx - m\), we get
\[
\int_{C_4} Mdx + Ndy = \int_{1}^{x_\alpha} \frac{-mx + m}{x^2 + (mx-m)^2} dx + \int_{1}^{x_\alpha} \frac{mxdx}{x^2 + (mx-m)^2}
\]
\[= \int_{1}^{x_\alpha} \frac{m}{x^2(1+m^2) - 2m^2x + m^2} dx
\]
\[= \frac{m}{1+m^2} \int_{1}^{x_\alpha} \frac{dx}{x^2 - \frac{2m^2}{1+m^2}x + \frac{m^4}{1+m^2}}
\]
\[= \frac{m}{1+m^2} \int_{1}^{x_\alpha} \frac{dx}{\left(x - \frac{m^2}{1+m^2}\right)^2 + \frac{m^2}{(1+m^2)^2}}
\]
\[= \frac{m}{|m|} \tan^{-1} \left( \frac{x + m^2x - m^2}{|m|} \right)_{1}^{x_\alpha}
\]
\[= \frac{m}{m} \tan^{-1} \left( \frac{x_\alpha + m^2x_\alpha - m^2}{|m|} \right) - \frac{m}{m} \tan^{-1} \left( \frac{1}{|m|} \right).
\]
Moreover, if \( x = \cos \frac{\theta}{2}, \ y = \sin \frac{\theta}{2}, \ \alpha \leq \theta \leq \beta \) are the parametric equations of \( C_5 \), we observe that
\[
\int_{C_5} Mdx + Ndy = \frac{2}{a} \int_{\alpha}^{\beta} \frac{\sin \frac{\theta}{2} - 1}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} \theta d\theta.
\]
The equation of \( C_6 \) is \( y = nx + 1 \), and so
\[
\int_{C_6} Mdx + Ndy = \int_{x_\beta}^{x_\alpha} -nx - 1 dx + \int_{x_\beta}^{x_\alpha} \frac{xndx}{x^2 + (nx + 1)^2}
= \int_{x_\beta}^{x_\alpha} \frac{dx}{x^2 + (nx + 1)^2}
= \frac{1}{n^2} \int_{x_\beta}^{x_\alpha} \frac{dx}{x + \frac{n}{1+n^2} x + \frac{1}{1+n^2}}
= \frac{1}{n^2} \int_{x_\beta}^{x_\alpha} \frac{dx}{(x + \frac{n}{1+n^2})^2 + \frac{1}{(1+n^2)^2}}
= \tan^{-1}(x + xn^2 + n) |_{x_\beta}^{x_\alpha}
= \tan^{-1}(x_\beta + x_\beta n^2 + n) - \tan^{-1}n.
\]
Finally,
\[
\int_{C_7} Mdx + Ndy = -2\pi
\]
and consequently,
\[
\frac{2}{a} \int_{\alpha}^{\beta} \frac{\sin \frac{\theta}{2} - 1}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} \theta d\theta = \int_{C_5} Mdx + Ndy
= -\sum_{i=1,i \neq 5}^{7} \int_{C_i} Mdx + Ndy
= \frac{\pi}{2} - m \tan^{-1}(\frac{x_\alpha + m^2 x_\alpha - m^2}{m}) + \tan^{-1}(\frac{1}{m})
- \tan^{-1}(x_\beta + x_\beta n^2 + n) + \tan^{-1}n
= \frac{\pi}{2} - \tan^{-1}(\frac{x_\alpha + m^2 x_\alpha - m^2}{m}) + \tan^{-1}(\frac{1}{m})
- \tan^{-1}(x_\beta + x_\beta n^2 + n) + \tan^{-1}n,
\]
and the result holds.

Taking \( \alpha = \beta \) in the previous theorem, we obtain the following relation on account of arc tangent function.

**Corollary 1.** Suppose that \( a \) is a positive real number, and \( 0 < \alpha < \frac{\pi}{2} \). If
\[
x_\alpha = \left(\frac{1}{1 + (\tan \alpha)^a}\right)^{1/a}, \ m = \frac{x_\alpha \tan \alpha}{x_\alpha - 1}, \ \text{and} \ n = \frac{x_\alpha \tan \alpha - 1}{x_\alpha},
\]
then
\[
\tan^{-1}\left(\frac{x_\alpha + m^2 x_\alpha - m^2}{m}\right) - \tan^{-1}(\frac{1}{m}) + \tan^{-1}(x_\alpha + x_\alpha n^2 + n) - \tan^{-1}n = \frac{\pi}{2}.
\]
For instance, using $\alpha = \pi/3$, and $a = 2$, we get $\tan^{-1}(2 - \sqrt{3}) = \frac{\pi}{12}$; and consequently, $\tan\frac{\pi}{12} = 2 - \sqrt{3}$.

**Theorem 2.** If $a$ and $b$ are two positive real numbers, then

$$
\int_{0}^{\pi/2} \cos^{a-1} \theta \sin^{a-1} \theta \left( \frac{2}{a} \sin^{2} \theta + \frac{2}{b} \cos^{2} \theta \right) d\theta = \pi/2.
$$

**Proof.** Let $C = \bigcup_{i=1}^{5} C_{i}$ be a positively oriented curve where $C_{1}, C_{2}$, and $C_{3}$ are the line segments as in the proof of Theorem 4.1. Also $C_{4}$ is a piece of the curve $x^{a} + y^{b} = 1$ from the point (1,0) to the point (0,1), and $C_{5}$ is the circle $x^{2} + y^{2} = r^{2}$ where $r > 0$ is sufficiently small so that it lies inside the closed curve $\bigcup_{i=1}^{4} C_{i}$. Putting

$$M(x, y) = \frac{-y}{x^{2} + y^{2}}, \quad N(x, y) = \frac{x}{x^{2} + y^{2}}$$

and then applying Green’s theorem, we get

$$\int_{C} Mdx + Ndy = \sum_{i=1}^{5} \int_{C_{i}} Mdx + Ndy = 0.$$

In the proof of the previous theorem, it was shown that $\sum_{i=1}^{3} \int_{C_{i}} Mdx + Ndy = \frac{3\pi}{2}$. Suppose that $x = \cos^{2} \theta$ and $y = \sin^{2} \theta$, $\theta \in [0, \frac{\pi}{2}]$ are the parametric equations of $C_{4}$. Since

$$\int_{C_{5}} Mdx + Ndy = -2\pi,$$

we have

$$\frac{\pi}{2} = \int_{C_{4}} Mdx + Ndy
= \int_{0}^{\pi/2} \left( -\sin^{2} \theta \right) \left( \frac{2}{a} \cos^{2} \theta + \frac{2}{b} \cos^{2} \theta \right) d\theta + \int_{0}^{\pi/2} \cos^{2} \theta \sin^{2} \theta \left( \frac{2}{a} \sin^{2} \theta + \frac{2}{b} \cos^{2} \theta \right) d\theta
= \int_{0}^{\pi/2} \cos^{a-1} \theta \sin^{a-1} \theta \left( \frac{2}{a} \sin^{2} \theta + \frac{2}{b} \cos^{2} \theta \right) d\theta,$$

and the result follows.

Note that Theorem 1 of [5], can be considered as a consequence of this theorem:

**Corollary 2.** If $n$ is a positive integer, then

$$\int_{0}^{\pi/2} \cos^{n-1} \theta \sin^{n-1} \theta \frac{d\theta}{\cos^{2n} \theta + \sin^{2n} \theta} = \frac{\pi}{2n}.$$
Another corollary to Theorem 1 run as follows.

**Corollary 3.** If \( n \) is an odd number, then

\[
\int_{0}^{2\pi} \frac{\cos^{n-1} \theta \sin^{n-1} \theta}{\cos^{2n} \theta + \sin^{2n} \theta} d\theta = \frac{2\pi}{n}.
\]

**Proof.** The integrand is an even function, and so in light of Corollary 2, we see that

\[
\frac{\pi}{2n} = \int_{0}^{\pi/2} \frac{\cos^{n-1} \theta \sin^{n-1} \theta}{\cos^{2n} \theta + \sin^{2n} \theta} d\theta
\]

\[
= \int_{-\pi/2}^{0} \frac{\cos^{n-1} \theta \sin^{n-1} \theta}{\cos^{2n} \theta + \sin^{2n} \theta} d\theta
\]

\[
= \int_{\pi/2}^{\pi} \frac{\cos^{n-1} (\theta + \pi) \sin^{n-1} (\theta + \pi)}{\cos^{2n} (\theta + \pi) + \sin^{2n} (\theta + \pi)} d\theta
\]

\[
= \int_{\pi/2}^{\pi} \frac{\cos^{n-1} \theta \sin^{n-1} \theta}{\cos^{2n} \theta + \sin^{2n} \theta} d\theta.
\]

(1)

Using the variables \( \theta + \frac{\pi}{2} \) and \( \theta + \pi \), respectively, instead of \( \theta \) in (1), gives us

\[
\int_{\pi}^{3\pi/2} \frac{\cos^{n-1} \theta \sin^{n-1} \theta}{\cos^{2n} \theta + \sin^{2n} \theta} d\theta = \frac{\pi}{2n}
\]

(2)

and

\[
\int_{3\pi/2}^{2\pi} \frac{\cos^{n-1} \theta \sin^{n-1} \theta}{\cos^{2n} \theta + \sin^{2n} \theta} d\theta = \frac{\pi}{2n}.
\]

(3)

Now, Corollary 2, along with the equalities (1), (2), and (3) imply the result.

**REFERENCES**


Received: May 29, 2007