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Evaluation of Certain Definite Integrals Involving Trigonometric Functions[1]

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Abstract

This paper is concerned with integrating of two classes of trigonometric functions using line integration. The first class, also gives us a formula involving arc tangent functions.

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Introduction

Besides its own interset, integration of trigonometric functions has important applications in many barnches of science. Except the usual ways, there are many other techniques to evaluate definite integrals. For instance, one way is to apply residue theorem [3], [4]. Using a recursion formula is another way [8]. It expresses the integral of a power of a function in terms of the integral of a lower power of the function. J. P. McCammond in 1999, gave a method for integrating polynomials in tangent and secant [7]. Some other techniques can be studied in [1], [2], and [6].

In this article, Green's theorem, one of the great, surprising theorems of calculus, is used to compute the integral of some rational functions of sine and

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cosine.

Green's Theorem. (Circulation, curl, or tangential form) The counterclockwise circulation of $\mathbf{F} = M\mathbf{i}+N\mathbf{j}$ around a simple closed curve C in the direction of its unit tangent vector $\mathbf{T}$, in the plane is the double integral of $(curl\mathbf{F}).\mathbf{k}$ over the region R enclosed by C;

$$\oint_C \mathbf{F}.\mathbf{T}\ ds = \oint_C Mdx + Ndy = \int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dx\ dy.$$

Main Results

Theorem 1. If a is a positive real number and $0 < \alpha \leq \beta < \pi/2$, then

$$\begin{aligned}\int_\alpha^\beta \frac{\sin^{\frac{2}{a}-1}\theta\cos^{\frac{2}{a}-1}\theta}{\cos^{\frac{4}{a}}\theta + \sin^{\frac{4}{a}}\theta}d\theta &= [\frac{\pi}{2} - tan^{-1}(\frac{x_\alpha + m^2x_\alpha - m^2}{m}) + tan^{-1}(\frac{1}{m}) \\ &- \tan^{-1}(x_\beta + x_\beta n^2 + n) + tan^{-1}n]\frac{a}{2},\end{aligned}$$

where

$$x_\alpha = (\frac{1}{1+(tan\alpha)^a})^{\frac{1}{a}}, \qquad x_\beta = (\frac{1}{1+(tan\beta)^a})^{\frac{1}{a}},$$

$$m = \frac{x_\alpha tan\alpha}{x_\alpha - 1}, \qquad n = \frac{x_\beta tan\beta - 1}{x_\beta}.$$

Proof. Let $C = \bigcup_{i=1}^{7} C_i$ be a positively oriented curve where $C_i, (i = 1, 2, 3)$ are the line segments from the point $(0, 1)$ to $(-1, 0)$ and $(-1, 0)$ to $(0, -1)$ and $(0, -1)$ to $(1, 0)$, respectively. By taking $y_\alpha = x_\alpha\ tan\ \alpha$ and $y_\beta = x_\beta\ tan\ \beta$, let C_4 be the line segment from the point $(1, 0)$ to (x_α, y_α) and C_5 be a piece of the curve $x^a + y^a = 1$ from the point (x_α, y_α) to the point (x_β, y_β). Also C_6 is the line segment from the point (x_β, y_β) to $(0, 1)$ and C_7 is the circle $x^2 + y^2 = r^2$ where $r > 0$ is sufficiently small so that C_7 lies inside the closed curve $\bigcup_{i=1}^{6} C_i$. Putting

$$M(x, y) = \frac{-y}{x^2 + y^2},\ N(x, y) = \frac{x}{x^2 + y^2}$$

and then applying Green's theorem, we conclude that

$$\oint_C Mdx + Ndy = \int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dA = 0,$$

where R is the region enclosed by C. Thus

$$\sum_{i=1}^{7}\int_{C_i} Mdx + Ndy = 0.$$

But

$$\begin{aligned}\int_{C_1} Mdx + Ndy &= \int_0^{-1} \frac{-1-x}{x^2+(1+x)^2}dx + \int_0^{-1} \frac{x}{x^2+(1+x)^2}dx \\ &= \int_0^{-1} \frac{-1}{x^2+(1+x)^2}dx\end{aligned}$$

and

$$\begin{aligned}\int_{C_2} Mdx + Ndy &= \int_{-1}^{0} \frac{x+1}{x^2+(x+1)^2}dx - \int_{-1}^{0} \frac{x}{x^2+(1+x)^2}dx \\ &= \int_{-1}^{0} \frac{dx}{(1+x)^2}.\end{aligned}$$

and

$$\begin{aligned}\int_{C_3} Mdx + Ndy &= \int_0^{1} \frac{1-x}{x^2+(x-1)^2}dx + \int_0^{1} \frac{x}{x^2+(x-1)^2}dx \\ &= \int_0^{1} \frac{dx}{x^2+(x-1)^2}.\end{aligned}$$

Furthermore,

$$\int_0^{-1} \frac{dx}{x^2+(1+x)^2} = \frac{1}{2}\int_0^{-1} \frac{dx}{(x+\frac{1}{2})^2+\frac{1}{4}} = tan^{-1}(-1) - tan^{-1}(1) = -\frac{\pi}{2}.$$

Therefore,

$$\sum_{i=1}^{3}\int_{C_i} Mdx + Ndy = \pi + \int_0^{1} \frac{dx}{x^2+(x-1)^2} = \pi - \int_0^{-1} \frac{dx}{x^2+(x+1)^2} = \frac{3\pi}{2}.$$

On the other hand, since C_4 is defined by $y = mx - m$, we get

$$\begin{aligned}\int_{C_4} Mdx + Ndy &= \int_1^{x_\alpha} \frac{-mx+m}{x^2+(mx-m)^2}dx + \int_1^{x_\alpha} \frac{mxdx}{x^2+(mx-m)^2} \\ &= \int_1^{x_\alpha} \frac{mdx}{x^2(1+m^2)-2m^2x+m^2} \\ &= \frac{m}{1+m^2}\int_1^{x_\alpha} \frac{dx}{x^2-\frac{2m^2}{1+m^2}x+\frac{m^2}{1+m^2}} \\ &= \frac{m}{1+m^2}\int_1^{x_\alpha} \frac{dx}{(x-\frac{m^2}{1+m^2})^2+\frac{m^2}{1+m^2}-\frac{m_4}{(1+m^2)^2}} \\ &= \frac{m}{1+m^2}\int_1^{x_\alpha} \frac{dx}{(x-\frac{m^2}{1+m^2})^2+\frac{m^2}{(1+m^2})^2} \\ &= \frac{m}{|m|}tan^{-1}\left(\frac{x+m^2x-m^2}{|m|}\right)|_1^{x_\alpha} \\ &= \frac{m}{|m|}tan^{-1}\left(\frac{x_\alpha+m^2x_\alpha-m^2}{|m|}\right) - \frac{m}{|m|}tan^{-1}(\frac{1}{|m|}).\end{aligned}$$

Moreover, if $x = \cos^{\frac{2}{a}}\theta,\ y = \sin^{\frac{2}{a}}\theta,\ \alpha \leq \theta \leq \beta$ are the parametric equations of C_5, we observe that

$$\int_{C_5} Mdx + Ndy = \frac{2}{a}\int_{\alpha}^{\beta} \frac{\sin^{\frac{2}{a}-1}\theta \cos^{\frac{2}{a}-1}\theta}{\cos^{\frac{4}{a}}\theta + \sin^{\frac{4}{a}}\theta} d\theta.$$

The equation of C_6 is $y = nx + 1$, and so

$$\begin{aligned}
\int_{C_6} Mdx + Ndy &= \int_{x_\beta}^{0} \frac{-nx-1}{x^2+(nx+1)^2}dx + \int_{x_\beta}^{0} \frac{xndx}{x^2+(nx+1)^2} \\
&= \int_{0}^{x_\beta} \frac{dx}{x^2(1+n^2)+2nx+1} \\
&= \frac{1}{1+n^2}\int_{0}^{x_\beta} \frac{dx}{x^2+\frac{2n}{1+n^2}x+\frac{1}{1+n^2}} \\
&= \frac{1}{1+n^2}\int_{0}^{x_\beta} \frac{dx}{(x+\frac{n}{1+n^2})^2+\frac{1}{(1+n^2)^2}} \\
&= tan^{-1}(x+xn^2+n)|_0^{x_\beta} \\
&= tan^{-1}(x_\beta+x_\beta n^2+n) - tan^{-1}n.
\end{aligned}$$

Finally,

$$\int_{C_7} Mdx + Ndy = -2\pi$$

and consequently,

$$\begin{aligned}
\frac{2}{a}\int_{\alpha}^{\beta} \frac{\sin^{\frac{2}{a}-1}\theta \cos^{\frac{2}{a}-1}\theta}{\cos^{\frac{4}{a}}\theta + \sin^{\frac{4}{a}}\theta} &= \int_{C_5} Mdx + Ndy \\
&= -\sum_{i=1, i\neq 5}^{7} \int_{C_i} Mdx + Ndy \\
&= \frac{\pi}{2} - \frac{m}{|m|}\tan^{-1}(\frac{x_\alpha + m^2 x_\alpha - m^2}{|m|}) + \frac{m}{|m|}\tan^{-1}(\frac{1}{|m|}) \\
&\quad - \tan^{-1}(x_\beta + x_\beta n^2 + n) + \tan^{-1} n \\
&= \frac{\pi}{2} - \tan^{-1}(\frac{x_\alpha + m^2 x_\alpha - m^2}{m}) + \tan^{-1}\frac{1}{m} \\
&\quad - \tan^{-1}(x_\beta + x_\beta n^2 + n) + \tan^{-1} n,
\end{aligned}$$

and the result holds.

Taking $\alpha = \beta$ in the previous theorem, we obtain the following relation on account of arc tangent function.

Corollary 1. Suppose that a is a positive real number, and $0 < \alpha < \frac{\pi}{2}$. If $x_\alpha = (\frac{1}{1+(\tan\alpha)^a})^{1/a}, m = \frac{x_\alpha \tan\alpha}{x_\alpha - 1}$, and $n = \frac{x_\alpha \tan\alpha - 1}{x_\alpha}$, then

$$\tan^{-1}(\frac{x_\alpha + m^2 x_\alpha - m^2}{m}) - \tan^{-1}\frac{1}{m} + \tan^{-1}(x_\alpha + x_\alpha n^2 + n) - \tan^{-1} n = \frac{\pi}{2}.$$

For instance, using $\alpha = \pi/3$, and $a = 2$, we get $tan^{-1}(2-\sqrt{3}) = \frac{\pi}{12}$; and consequently, $tan\frac{\pi}{12} = 2-\sqrt{3}$.

Theorem 2. If a and b are two positive real numbers, then

$$\int_0^{\pi/2} \frac{\cos^{\frac{2}{a}-1}\theta \sin^{\frac{2}{a}-1}\theta(\frac{2}{a}\sin^2\theta + \frac{2}{b}\cos^2\theta)}{\cos^{\frac{4}{a}}\theta + \sin^{\frac{4}{b}}\theta} d\theta = \pi/2.$$

Proof. Let $C = \bigcup_{i=1}^{5} C_i$ be a positiviely oriented curve where C_1, C_2, and C_3 are the line segments as in the proof of Theorem 4.1. Also C_4 is a piece of the curve $x^a + y^b = 1$ from the point (1,0) to the point (0,1), and C_5 is the circle $x^2 + y^2 = r^2$ where $r > 0$ is sufficiently small so that it lies inside the closed curve $\bigcup_{i=1}^{4} C_i$. Putting

$$M(x,y) = \frac{-y}{x^2+y^2}, N(x,y) = \frac{x}{x^2+y^2}$$

and then applying Green's theorem, we get

$$\int_C Mdx + Ndy = \sum_{i=1}^{5} \int_{C_i} Mdx + Ndy = 0.$$

In the proof of the previous theorem, it was shown that $\sum_{i=1}^{3} \int_{C_i} Mdx + Ndy = \frac{3\pi}{2}$. Suppose that $x = \cos^{\frac{2}{a}}\theta$ and $y = \sin^{\frac{2}{b}}\theta$, $\theta \in [0, \frac{\pi}{2}]$ are the parametric equations of C_4. Since

$$\int_{C_5} Mdx + Ndy = -2\pi,$$

we have

$$\begin{aligned}
\frac{\pi}{2} &= \int_{C_4} Mdx + Ndy \\
&= \int_0^{\frac{\pi}{2}} \frac{(-\sin^{\frac{2}{b}}\theta)(\frac{2}{a}\cos^{\frac{2}{a}-1}\theta)(-\sin\theta)}{\cos^{\frac{4}{a}}\theta + \sin^{\frac{4}{b}}\theta} d\theta + \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{2}{a}}\theta(\frac{2}{b}\sin^{\frac{2}{b}-1}\theta\cos\theta)d\theta}{\cos^{\frac{4}{a}}\theta + \sin^{\frac{4}{b}}\theta} \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{2}{a}-1}\theta \sin^{\frac{2}{b}-1}\theta(\frac{2}{a}\sin^2\theta + \frac{2}{b}\cos^2\theta)}{\cos^{\frac{4}{a}}\theta + \sin^{\frac{4}{b}}\theta} d\theta,
\end{aligned}$$

and the result follows.

Note that Theorem 1 of [5], can be considered as a consequence of this theorem:

Corollary 2. If n is a positive integer, then

$$\int_0^{\pi/2} \frac{\cos^{n-1}\theta \sin^{n-1}\theta}{\cos^{2n}\theta + \sin^{2n}\theta} d\theta = \frac{\pi}{2n}.$$

Another corollary to Theorem 1 run as follows.

Corollary 3. If n is an odd number, then

$$\int_0^{2\pi} \frac{\cos^{n-1}\theta \sin^{n-1}\theta}{\cos^{2n}\theta + \sin^{2n}\theta} d\theta = \frac{2\pi}{n}.$$

Proof. The integrand is an even function, and so in light of Corollary 2, we see that

$$\begin{aligned}
\frac{\pi}{2n} &= \int_0^{\pi/2} \frac{\cos^{n-1}\theta \sin^{n-1}\theta}{\cos^{2n}\theta + \sin^{2n}\theta} d\theta \\
&= \int_{-\pi/2}^{0} \frac{\cos^{n-1}\theta \sin^{n-1}\theta}{\cos^{2n}\theta + \sin^{2n}\theta} d\theta \\
&= \int_{\pi/2}^{\pi} \frac{\cos^{n-1}(\theta+\pi) \sin^{n-1}(\theta+\pi)}{\cos^{2n}(\theta+\pi) + \sin^{2n}(\theta+\pi)} d\theta \\
&= \int_{\pi/2}^{\pi} \frac{\cos^{n-1}\theta \sin^{n-1}\theta}{\cos^{2n}\theta + \sin^{2n}\theta} d\theta. \qquad (1)
\end{aligned}$$

Using the variables $\theta + \frac{\pi}{2}$ and $\theta + \pi$, respectively, instead of θ in (1), gives us

$$\int_\pi^{3\pi/2} \frac{\cos^{n-1}\theta \sin^{n-1}\theta}{\cos^{2n}\theta + \sin^{2n}\theta} d\theta = \frac{\pi}{2n} \qquad (2)$$

and

$$\int_{3\pi/2}^{2\pi} \frac{\cos^{n-1}\theta \sin^{n-1}\theta}{\cos^{2n}\theta + \sin^{2n}\theta} d\theta = \frac{\pi}{2n}. \qquad (3)$$

Now, Corollary 2, along with the equalities (1), (2), and (3) imply the result.

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