

Some Bounds for the Product of Singular Values

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Abstract

Let A be an $n \times n$ complex matrix with $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_n(A)$ and let $1 \leq k < l \leq n$. Bounds for $\sigma_1(A) \dots \sigma_k(A)$, $\sigma_k(A) \dots \sigma_l(A)$ and $\sigma_{n-k+1}(A) \dots \sigma_n(A)$, involving k , n , $r_i(A)$ ($c_i(A)$), α_i and $\det A$ where $r_i(c_i(A))$ is the Euclidean norm of the i -th row (column) of A and α_i 's are positive real numbers such that $\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = n$, are presented.

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1 Introduction and Preliminaries

Let A be an $n \times n$ complex matrix. The singular values of A , denoted $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_n(A)$, are the nonnegative square roots of the eigenvalues of the positive semi-definite matrix A^*A .

It is well known that

$$\sigma_1^2(A) + \sigma_2^2(A) + \dots + \sigma_n^2(A) = \|A\|_F^2 \quad (1.1)$$

and

$$\sigma_1(A) \sigma_2(A) \dots \sigma_n(A) = |\det A| \quad (1.2)$$

where $\|A\|_F$ and $\det A$ denote the Frobenius norm of A and the determinant of A , respectively.

Merikoski and Virtanen [2] obtained bounds for eigenvalue using trace and determinant. Rojo [3] defined B matrix. Wang and Zhang [4] have established

some inequalities for the eigenvalues of the product of positive semidefinite Hermitian matrices. The singular values are very useful in the study of problems in different areas (see [HJ, Chapter 3], [5], and their references). This paper presents a different approach for finding the bounds for the product of singular values.

In Section 2, we have obtained bounds for products of singular values using bounds for eigenvalues in [2] and (1.1) and (1.2) equalities. In Section 3, we have found bounds for products of singular values using row (column) norm and determinant.

Firstly, we give some preliminaries related to our study.

To minimize the numerical round-off errors in solving the system $Ax = b$, it is normally convenient that the rows of A be properly scaled before the solution procedure begins. One way is to premultiply by the diagonal matrix

$$D = \text{diag} \left\{ \frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, \dots, \frac{\alpha_n}{r_n(A)} \right\}, \quad (1.3)$$

where $r_i(A)$ is the Euclidean norm of the i -th row of A and $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive real numbers such that

$$\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = n. \quad (1.4)$$

Clearly, the Euclidean norm of the coefficient matrix $B = DA$ of the scaled system is equal to \sqrt{n} and if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ then each row of B is a unit vector in the Euclidean norm. Also, we can define $B = AD$,

$$D = \text{diag} \left\{ \frac{\alpha_1}{c_1(A)}, \frac{\alpha_2}{c_2(A)}, \dots, \frac{\alpha_n}{c_n(A)} \right\}, \quad (1.5)$$

where $c_i(A)$ is the Euclidean norm of the i -th column of A . Again, $\|B\|_E = \sqrt{n}$ and if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ then each column of B is a unit vector in the Euclidean norm.

Theorem 1 [4] *Let $A, B \in \mathbb{C}^{n \times n}$ and let $1 \leq i_1 < \dots < i_k \leq n$. Then*

$$\prod_{t=1}^k \sigma_{i_t}(AB) \leq \prod_{t=1}^k \sigma_{i_t}(A) \sigma_t(B). \quad (1.6)$$

Theorem 2 [1] *Let $A \in \mathbb{C}^{p \times n}$, $B \in \mathbb{C}^{n \times m}$ and let $1 \leq i_1 < \dots < i_k \leq n$. Then*

$$\prod_{t=1}^k \sigma_{i_t}(AB) \geq \prod_{t=1}^k \sigma_{i_t}(A) \sigma_{n-t+1}(B). \quad (1.7)$$

2 Bounds for product of singular values using norm and determinant

We will give bounds that we obtained for singular values as a result of some bounds for eigenvalue which is obtained in [2] where A be a square matrix with singular values

$$\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_n(A) > 0.$$

Corollary 3 *Let $1 \leq k \leq n$. Then*

$$\begin{aligned} (\sigma_{n-k+1}(A) \dots \sigma_n(A))^{2/k} &\leq \frac{\sigma_{n-k+1}^2(A) + \dots + \sigma_n^2(A)}{k} \leq \frac{\|A\|_F^2}{n} \\ &\leq \frac{\sigma_1^2(A) + \dots + \sigma_k^2(A)}{k} \end{aligned} \tag{2.1}$$

and

$$\sigma_{n-k+1}(A) \dots \sigma_n(A) \geq \left(\frac{n-k}{\|A\|_F^2} \right)^{(n-k)/2} |\det A|. \tag{2.2}$$

Corollary 4 *Let $1 \leq k \leq l \leq n$. Then*

$$\left(\frac{k-1}{\|A\|_F^2} \right)^{(k-1)/2} |\det A|^{l/n} \leq \sigma_k(A) \dots \sigma_l(A) \leq \left(\frac{\|A\|_F^2}{l} \right)^{l/2} \left(\frac{1}{|\det A|} \right)^{(k-1)/n}. \tag{2.3}$$

3 Bounds for product of singular values using row (column) norm and determinant

Since the matrices PA , AP and A have the same singular values for any permutation matrix P , we assume for this section, without loss of generality, that the rows and columns of A are such that

$$r_1(A) \leq r_2(A) \leq \dots \leq r_n(A) \tag{3.1}$$

$$c_1(A) \leq c_2(A) \leq \dots \leq c_n(A) \tag{3.2}$$

Let

$$0 < \alpha_n \leq \dots \leq \alpha_2 \leq \alpha_1 \tag{3.3}$$

for α_i 's in (1.4).

Theorem 5 *Let $1 < k \leq n$. Then*

$$\sigma_{n-k+1}(A) \dots \sigma_n(A) \leq \prod_{i=n-k+1}^n \frac{\min\{c_i(A), r_i(A)\}}{\alpha_i} \quad (3.4)$$

where $r_i(A)$ ($c_i(A)$)'s and α_i 's be as in (3.1) ((3.2)) and (3.3), respectively.

Proof. We write

$$(\sigma_{n-k+1}(B) \dots \sigma_n(B))^{2/k} \leq 1 \quad (3.5)$$

for matrix $B = DA$ which is defined in Section 1 from (2.1). Taking $i_1 = n - k + 1$ and $i_k = n$ in (1.7) and applying matrix $B = DA$ which is defined in Section 1, we obtain

$$\sigma_{n-k+1}(B) \dots \sigma_n(B) \geq \sigma_{n-k+1}(D) \dots \sigma_n(D) \sigma_{n-k+1}(A) \dots \sigma_n(A).$$

Consequently, we have

$$\sigma_{n-k+1}(A) \dots \sigma_n(A) \leq \prod_{i=n-k+1}^n \frac{r_i}{\alpha_i}$$

from (3.5).

The corresponding result for columns can be obtained by considering matrix $B = AD$ which is defined in Section 1. ■

Theorem 6 *Let $1 < k \leq n$. Then*

$$\sigma_1(A) \dots \sigma_k(A) \geq \prod_{i=1}^{n-k} \frac{\alpha_i}{r_i(A)} \left(\frac{n-k}{n}\right)^{(n-k)/2} |\det A| \quad (3.6)$$

where $r_i(A)$'s and α_i 's be as in (3.1) and (3.3), respectively.

Proof. We write

$$\sigma_{n-k+1}(B) \dots \sigma_n(B) \geq \left(\frac{n-k}{n}\right)^{(n-k)/2} \prod_{i=1}^n \frac{\alpha_i}{r_i} |\det A|$$

for matrix $B = DA$ which is defined in Section 1 from inequality (2.2). Taking $i_1 = n - k + 1$ and $i_k = n$ in (1.6) and applying matrix $B = DA$ which is defined in Section 1, we obtain

$$\sigma_{n-k+1}(B) \dots \sigma_n(B) \leq \sigma_{n-k+1}(D) \dots \sigma_n(D) \sigma_1(A) \dots \sigma_k(A).$$

Consequently, we have (3.6). ■

Theorem 7 *Let $1 \leq k < l \leq n$. Then*

$$\prod_{i=1}^{l-k+1} \frac{c_i(A)}{\alpha_i} \left(\frac{k-1}{n}\right)^{(k-1)/2} \left(\prod_{i=1}^n \frac{\alpha_i}{c_i(A)} |\det A|\right)^{l/n} \leq \sigma_k(A) \dots \sigma_l(A)$$

$$\leq \prod_{i=n-l+k}^n \frac{c_i(A)}{\alpha_i} \left(\frac{n}{l}\right)^{l/2} \left(\frac{\prod_{i=1}^n \frac{c_i(A)}{\alpha_i}}{|\det A|}\right)^{(k-1)/n} \tag{3.7}$$

where $c_i(A)$'s and α_i 's be as in (3.2) and (3.3), respectively.

Proof. First, we write (2.3) for matrix B which is defined in Section 1. Taking $i_1 = k, i_k = l$ in (1.6) and applying matrix $B = AD$ which is defined in Section 1, we obtain (3.7). ■

Theorem 8 *Let $1 \leq k < l \leq n$. Then*

$$\sigma_1(A) \dots \sigma_{l-k+1}(A) \geq \prod_{i=k}^l \frac{r_i(A)}{\alpha_i} \left(\frac{k-1}{n}\right)^{(k-1)/2} \left(\prod_{i=1}^n \frac{\alpha_i}{r_i(A)} |\det A|\right)^{l/n} \tag{3.8}$$

where $r_i(A)$'s and α_i 's be as in (3.1) and (3.3), respectively.

Proof. Write inequality (2.3) for matrix B which is defined in Section 1. Taking $i_1 = k, i_k = l$ in (1.6) and applying matrix $B = DA$ which is defined in Section 1. ■

Theorem 9 *Let $1 < k \leq n$. Then*

$$\sigma_{n-k+1}(A) \dots \sigma_n(A) \geq \prod_{i=k+1}^n \frac{\alpha_i}{c_i(A)} \left(\frac{n-k}{n}\right)^{(n-k)/2} |\det A| \tag{3.9}$$

where $c_i(A)$'s and α_i 's be as in (3.2) and (3.3), respectively.

Proof. Write inequality (2.2) for matrix B which is defined in Section 1. Taking $i_1 = n - k + 1, i_k = n$ in (1.6) and applying matrix $B = AD$ which is defined in Section 1, we have inequality (3.9). ■

Example 10 Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 2 \end{bmatrix}$. Singular values matrix of A are $\sigma_1(A) = 6.673$, $\sigma_2(A) = 1.949$, $\sigma_3(A) = 1.181$ and $\sigma_4(A) = 0.521$. In the following, the best bounds are underlined.

$k \setminus l$	1	2	3	4
1	σ_1	$\sigma_1\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3\sigma_4$
2		σ_2	$\sigma_2\sigma_3$	$\sigma_2\sigma_3\sigma_4$
3			σ_3	$\sigma_3\sigma_4$
4				σ_4

For product of singular values, we have the following values:

1. The values of product of singular values:

$k \setminus l$	1	2	3	4
1	6.673	13.008	15.362	8
2		1.949	2.302	1.199
3			1.181	0.615
4				0.521

2. Lower bounds for product of singular values:

(3.6)

$k \setminus l$	1	2	3	4
1		0.684	<u>1.965</u>	8

(3.7)

$k \setminus l$	1	2	3	4
1		<u>0.885</u>	1.06	8
2			0.157	0.189
3				0.056

(3.8)

$k \setminus l$	1	2	3	4
1		0.766	1.905	8

(3.9)

$k \setminus l$	1	2	3	4
1				8
2			0.189	
3			0.056	

3. Upper bounds for product of singular values:

(3.4)

$k \setminus l$	1	2	3	4
1				<u>417.607</u>
2				<u>205.174</u>
3				<u>71.376</u>

(3.7)

$k \setminus l$	1	2	3	4
1		142.752	315.886	499.133
2			308.846	576.639
3				563.788

We have seen that some bounds for $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ are better than those for α_i 's ($i = 1, \dots, n$) such that $\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = n$ while some bounds are worse. For example, for $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$, upper bound (3.4) is better in the form

$k \setminus l$	1	2	3	4
1				122.963
2				46.476
3				14.697

while lower bound (3.6) is worse in the form

$k \setminus l$	1	2	3	4
1		0.478	1.152	8

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