Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 25, 1221 - 1225

An Euler Product for $(\frac{\pi}{k})\cot(\frac{\pi}{k})$

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Abstract

We use the partial fraction decomposition of the cotangent function in order to obtain its Euler product representation.

Mathematics Subject Classification: 11M41, 14G10

Keywords: Dirichlet Series, Euler Product, Infinite Product, Multiplicative Function

1 Introduction

The Euler Product identity

$$\prod_p (1 - \frac{f(p)}{p^s})^{-1} = \sum_{n \geq 1} \frac{f(n)}{n^s}$$

in which the product is extended over all primes p, is valid for any completely multiplicative function $f(n)$ for which the series converges absolutely [4]. If f is non principal Dirichlet character $mod(k)$ (integer parameter $k \geq 3$), the both the Dirichlet series and its Euler product are absolutely convergent for each fixed s in the half-plane $\Re e(s) > 1$, and each is uniformly convergent on compact subsets of this half-plane, so each represents an analytic function of s in this half-plane. The series also converges at $s = 1$ so the function it represents is continuous at $s = 1$. Consequently the same is true of the infinite product. Hence for this choice of f the Euler product identity is also valid when $s = 1$. Putting $s = 1$ we find:

$$\prod_p (\frac{p}{p - f(p)}) = \sum_{m \geq 1} \frac{f(m)}{m} \qquad (1)$$

2 First Case

Let us define the function $F(m)$ as:

$$F(m) = (-1)^r \text{ if } \quad m \equiv (-1)^r (modk), \quad F(m) = 0 \text{ otherwise.}$$

If $k = 3, 4$ or 6; F is a non principal Dirichlet character $modk$.
Note that:
$F(1) = 1$, $F(m) = -1$ if $m = nk - 1$, $F(m) = 1$ if $m = nk + 1$ and $F(m) = 0$; otherwise.
Taking $f = F$ in (1) gives

$$\sum_{m \geq 1} \frac{F(m)}{m} = 1 + \sum_{n \geq 1} (\frac{1}{nk+1} - \frac{1}{nk-1}),$$

now we use the partial fraction decomposition of the cotangente,

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{m=-\infty, m \neq 0} (\frac{1}{z+m} - \frac{1}{m}) = \frac{1}{z} + \sum_{m \geq 1} (\frac{1}{m+z} - \frac{1}{m-z}) \qquad (2)$$

which is valid for $|z| < 1$. Taking $z = \frac{1}{k}$ we obtain

$$\sum_{m \geq 1} \frac{F(m)}{m} = (\frac{\pi}{k}) \cot(\frac{\pi}{k}).$$

consequently, if $k = 3, 4$ or 6 we have the following Euler product representation:

$$\prod_p (\frac{p}{p - F(p)}) = (\frac{\pi}{k}) \cot(\frac{\pi}{k}).$$

Note that $F(p) \neq 0$ if $p \equiv \pm 1 (modk)$, which implies k divides $p + 1$ or $p - 1$. this can not occur if $k \geq 2p$, hence $F(p) = 0$ if $p \leq \frac{k}{2}$. Therefore only primes greater than $\frac{k}{2}$ that are congruent to $\pm 1 (modk)$ need be taken in the infinite product. But if $p = nk \pm 1$, then $p - f(p) = nk$, which is the multiple of k nearest to p. Therefore we have proved the following theorem.

Theorem 2.1 *Let $M_k(p)$ denote the multiple of k nearest to p. Then for $k = 3, 4$ or 6 we have for each p prime*

$$\prod_{p > \frac{k}{2}} (\frac{p}{M_k(p)}) = (\frac{\pi}{k}) \cot(\frac{\pi}{k}) \qquad (3)$$

Example 2.2 *When $k = 4$ obtain a formula of Euler*

$$\frac{3}{4} \frac{5}{4} \frac{7}{8} \frac{11}{12} \frac{13}{12} \frac{17}{16} \frac{19}{20} \frac{23}{24} \frac{29}{28} \frac{31}{32} \cdots = \frac{\pi}{4}$$

multiplying both members by $\frac{2}{4}$ *we get*

$$\frac{2}{4}\frac{3}{4}\frac{5}{4}\frac{7}{8}\frac{11}{12}\frac{13}{12}\frac{17}{16}\frac{19}{20}\frac{23}{24}\frac{29}{28}\frac{31}{32}\cdots=\frac{\pi}{8},$$

when $k=3$ *or* 6*, we obtain two more formulas of Euler*

$$\text{when } k=3,\quad \frac{2}{3}\frac{5}{6}\frac{7}{6}\frac{11}{12}\frac{13}{12}\frac{17}{18}\frac{19}{18}\frac{23}{24}\frac{29}{30}\frac{31}{30}\cdots=\frac{\pi}{3\sqrt{3}}$$

$$\text{when } k=6,\quad \frac{5}{6}\frac{7}{6}\frac{11}{12}\frac{13}{12}\frac{17}{18}\frac{19}{18}\frac{23}{24}\frac{29}{30}\frac{31}{30}\frac{37}{36}\cdots=\frac{\pi}{2\sqrt{3}}.$$

3 Second Case

Unfortunately if $k\neq 3,4$ or 6; F is not a multiplicative function. But when $k=5$ we may take f as the following non principal Dirichlet character $mod5$. $f(m)=(-1)^r$ if $m\equiv(-1)^r(mod5)$; $f(m)=\pm i$ if $m\equiv\pm 2(mod5)$, $i^2=-1$, $f(m)=0$ otherwise.

Now we evaluate the absolute value of the product in the Euler product identity:

$$|\sum_{m\geq 1}\frac{f(m)}{m^s}|=|\prod_p(\frac{p^s}{p^s-f(p)})|.$$

Since both members of the last equality are absolutely convergent, with $s>1$ in real line, we have:

$$|\prod_p(\frac{p^s}{p^s-f(p)})|=\prod_{p\equiv\pm 1(mod5)}\frac{p^s}{|p^s-f(p)|}\prod_{p\equiv\pm 2(mod5)}\frac{p^s}{|p^s-f(p)|}|\frac{5^s}{5^s-f(5)}|;$$

moreover, from the definition of f:
if $p\equiv\pm 1(mod5)$ then $f(p)=\pm 1$ and $|p^s-f(p)|^2=(p^s-f(p))^2$, $f(5)=0$,
if $p\equiv\pm 2(mod5)$ then $|p^s-f(p)|^2=|p^s\pm i|^2=p^{2s}+1$, then the infinite product gives

$$|\prod_p(\frac{p^s}{p^s-f(p)})|^2=(\prod_{p\equiv\pm 1(mod5)}\frac{p^s}{p^s-f(p)})^2\prod_{p\equiv\pm 2(mod5)}\frac{p^{2s}}{p^{2s}+1}$$

$$=\prod_{p\equiv\pm 1(mod5)}\frac{p^{2s}+1}{(p^s-f(p))^2}\prod_{p\,prime}(\frac{p^{2s}}{p^{2s}+1})(\frac{5^{2s}+1}{5^{2s}})$$

$$=\prod_{p\equiv\pm 1(mod5)}\frac{p^{2s}+1}{(p^s-f(p))^2}(\frac{\zeta(4s)}{\zeta(2s)})(\frac{5^{2s}+1}{5^{2s}})$$

where ζ is the Riemann zeta function.
Because the Dirichlet series converges at $s = 1$ and is continuous there, the infinite product also converges at $s = 1$.
When $s = 1$ we find

$$|\sum_{m\geq 1}\frac{f(m)}{m}|^2 = \prod_{p\equiv\pm 1(mod5)}\frac{p^2+1}{(p-f(p))^2}\cdot(\frac{\pi^2}{15}\cdot\frac{26}{25}) \qquad (4)$$

on the other hand, by equation (2)

$$|\sum_{m\geq 1}\frac{f(m)}{m}| = |1+\sum_{n\geq 1}(\frac{1}{5n-1}-\frac{1}{5n-1})+\frac{i}{2}+\sum_{n\geq 1}(\frac{i}{5n+2}-\frac{i}{5n-2})|$$

$$= \frac{\pi}{5}|\cot(\frac{\pi}{5})+i\cot(\frac{2\pi}{5})| = \frac{\pi\sqrt{2}}{5} \qquad (5)$$

moreover from the first case we know that if $p\equiv\pm 1(mod5)$ then $p-f(p)$ is the multiple of 5 nearest to p.
Combining the last result with equations (4) and (5) we obtain the following theorem

Theorem 3.1 *Let $M_5(p)$ denote the multiple of k nearest to p, then*

$$\prod_{p\equiv\pm 1(mod5)}(\frac{p^2+1}{M_5(p)^2}) = \frac{15}{13}.$$

developing the first few terms, we find

$$\frac{11^2+1}{10^2}\cdot\frac{19^2+1}{20^2}\cdot\frac{29^2+1}{30^2}\cdot\frac{31^2+1}{30^2}\cdot\frac{41^2+1}{40^2}\cdot\frac{59^2+1}{60^2}\cdot\frac{61^2+1}{60^2}\cdot\frac{71^2+1}{70^2}\cdots = \frac{15}{13}$$

which is an approximation to the infinite product

$$\prod_{p>\frac{5}{2}}(\frac{p}{M_5(p)}) \quad p \text{ prime.}$$

Finally, two open problems remain:

(i) In general, to find out to what limit the following infinite product converges to

$$\prod_{p>\frac{k}{2}}(\frac{p}{M_k(p)}) \quad p \text{ prime, forall } k \text{ integer, } k\geq 3.$$

(ii) For what values of k is the equation (3) true?

ACKNOWLEDGEMENTS. I am grateful to the department of mathematics of the Universidad Autónoma Metropolitana-Iztapalapa for the valuable help to write this paper.

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Received: April 17, 2007