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Abstract

Stochastic integro partial differential equations of the form;

\[ du(x,t) = \sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial x_i^2} \, dt + F(u(x,t),x,t)dt \]

\[ + \int_0^t K(t-\theta)u(x,\theta)d\theta \, dt + [f(t)u(x,t) + g(x,t)]dW(t), \]

are considered, where \( \{W(t) : t \geq 0\} \) is a standard one-dimensional Wiener process and the kernel K decreases to zero non-exponentially. The behavior of solutions and their convergence to zero are studied. It is proved under suitable conditions that

\[ \lim_{t \to \infty} \frac{u(x,t)}{K(t)} = \infty, \]

almost surely.

The considered stochastic integro partial differential equations arise if we consider the Black-Scholes market consists of a bank account or a bond and a stock. These stochastic models can also applied to population dynamics in biology.

Mathematics Subject Classifications: 34D20, 60H10, 65H20, 45N05

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1. Introduction

In this paper, we consider the stochastic integro partial differential equations of the form

\[ du(x,t) = [\nabla^2 u(x,t) + F(u(x,t),x,t)]dt \]
\[
+ \int_0^t K(t - \theta)u(x, \theta)d\theta dt + \left[ f(t)u(x, t) + g(x, t) \right] dW(t), \quad (1.1)
\]
with the initial condition
\[
u(x, 0) = \varphi(x) \quad (1.2)
\]
where \( \varphi \) is a bounded continuous function on the n-dimensional Euclidean space \( R^n, x \in R^n, \nabla^2 = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \) and \( \{ W(t) : t \geq 0 \} \) is a standard Wiener process on a complete filtered probability space \( \{ \Omega, F, F_t, P \} \), where the filtration is the natural one, viz. \( F_t = \sigma \{ W(s) : 0 \leq s \leq t \} \). The almost sure events considered in this paper are always \( P \)-almost sure, (a.s). The functions \( F, K, f \) and \( g \) satisfy the following conditions:
\( A_1 : F \) is continuous on \( \mathbb{R} \times R^n \times [0, \infty) \), \( \mathbb{R} = (-\infty, \infty) \) and satisfies the Lipschitz condition;
\[
|F(u, x, t) - F(v, x, t)| \leq L|u - v|,
\]
for all \( (u, x, t), (v, x, t) \in \mathbb{R} \times R^n \times [0, \infty) \) and for some positive constant \( L \),
\( A_2 : K, f \) are continuous on \( [0, \infty) \) and \( g \) is continuous on \( R^n \times [0, \infty) \).
In section 2, we shall give a suitable representation of the sample paths of the considered problem. The existence and uniqueness of the solutions are treated for more general equations, (see [1-6]). Here we are interested in the stability of solutions. In section 3, we study the phenomenon of the non-exponential stability, (see [7]).
Many special cases of equation (1.1) have several applications to population dynamics in biology and also to Black - Scholes market, which consists of a bank account or a bond and a stock.

2. Representation of solutions

Let \( C(R^n \times (0, T]) \) be the set of all continuous functions on
\[
R^n \times (0, T], T < \infty. \text{ If } u, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x_i^2} \in C(R^n \times (0, T]),
\]
\( i = 1, \ldots, n \), for almost all \( w \in \Omega \) where \( u \) is a solution of the Cauchy problem (1.1), (1.2), for almost all \( w \in \Omega \), then:
\[
u(x, t) = \int_{R^n} Z(x - y, t)\varphi(y)dy
+ \int_0^t \int_{R^n} Z(x - y, t - s)F(u(y, s), y, s)dy ds
+ \int_0^t \int_{R^n} Z(x - y, t - s)\int_0^s K(s - \theta)u(y, \theta)d\theta dy ds
+ \int_0^t \int_{R^n} Z(x - y, t - s)[f(s)u(y, s) + g(y, s)] dy dW(s), \quad (2.1)
\]
where
\[ Z(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{1}{4t} \sum_{i=1}^{n} x_i^2 \right). \]

If \( u \in C(R^n \times [0,T]) \) satisfies equation (2.1) for almost all \( w \in \Omega \), then \( u \) is called a mild solution of the Cauchy problem (1.1),(1.2).

If \( u_1 \) and \( u_2 \) are two solutions, a.s., of (1.1), (1.2) then assumptions \( A_1, A_2 \) and the properties of the Wiener process \( W(t) \) lead to the following fact:

\[
\Lambda(t) \leq L^2T \int_0^t \int_{R^n} Z(y,s)\Lambda(s)dy \, ds + T^2 \int_0^t \int_s^t \int_{R^n} Z(y,s)K^2(s-\theta)\Lambda(\theta)dy \, d\theta \, ds + \int_0^t \int_{R^n} Z(y,s)f^2(s)\Lambda(s)dy \, ds,
\]

where:
\[
\Lambda(t) = \sup_{x \in R^n} E\left[\{u_1(x,t) - u_2(x,t)\}^2\right]
\]
and \( E(X) \) is the expectation of \( X \). Since \( \int_{R^n} Z(y,s)dy = 1 \), it follows that
\[
\Lambda(t) \leq c \int_0^t \Lambda(s)ds,
\]
for some positive constant \( c \). This proves the uniqueness of the mild solutions.

Let us try to find the solution of the considered problem in the form
\[
u(x,t) = V(t)\psi(x,t),
\]
where \( \{V(t) : t \geq 0\} \) is the stochastic process described by the following equation
\[
dV(t) = -aV(t) + f(t) \, V(t) dW(t), \quad (2.2)
\]
\[ V(0) = 1 \quad (2.3) \]

where \( a \in R \).

The solution of (2.2), (2.3) is given by
\[
V(t) = \exp\left[\int_0^t f(s)dW(s) - \int_0^t \frac{1}{2}f^2(s) + a\right]ds.
\]

Let us assume that
\[
\{\psi(x,t) : (x,t) \in R^n \times [0,\infty)\}
\]
is the stochastic process described by the following equation:
\[
d\psi(x,t) = \nabla^2\psi(x,t)dt + A(x,t)dt + B(x,t)dW(t), \quad (2.4)
\]
\[ \psi(x, 0) = \varphi(x), \quad (2.5) \]

where A and B are chosen such that \( u(x, t) = V(t) \psi(x, t) \) satisfies equation (1.1).

We have:

\[
\begin{align*}
du(x, t) &= V(t)d\psi(x, t) + \psi(x, t)dV(t) \\
&\quad + f(t)V(t)B(x, t)dt
\end{align*}
\]

Using (2.2), (2.3), (2.4) and (2.5), we find that it is suitable to choose A and B as follows:

\[
\begin{align*}
A(x, t) &= V^{-1}(t)\eta(x, t), \\
B(x, t) &= V^{-1}(t)g(x, t),
\end{align*}
\]

where

\[
\eta(x, t) = G(u(x, t), x, t) + \int_0^t K(t-\theta)u(x, \theta)d\theta - f(t)g(x, t), \quad (2.6)
\]

\[
G(u, x, t) = F(u, x, t) + au.
\]

Using (2.4), (2.5) and (2.6), one gets the following representation of \( u \):

\[
\begin{align*}
u(x, t) &= V(t)\int_{\mathbb{R}^n} Z(x - y, t)\varphi(y)dy \\
&\quad + \int_0^t \int_{\mathbb{R}^n} Z(x - y, t - s)H(s, t)\eta(y, s)dy ds \\
&\quad + \int_0^t \int_{\mathbb{R}^n} Z(x - y, t - s)H(s, t)g(y, s)dy dW(s), \quad (2.7)
\end{align*}
\]

where \( H(s, t) = V(t)V^{-1}(s) \).

Now we prove that the stochastic process \( \{\eta(x, t) : (x, t) \in \mathbb{R}^n \times [0, \infty)\} \) satisfies a nonlinear Riemann integral equation with random kernels.

**Lemma 2.1.** If the stochastic process \( \{\eta(x, t) : (x, t) \in \mathbb{R}^n \times [0, \infty)\} \) is given by (2.6), then

\[
\begin{align*}
\eta(x, t) &= \int_0^t S(\theta, t)\eta^*(x, \theta, t)d\theta \\
&\quad + \int_0^t K(t - \theta)h(x, \theta)d\theta + G(u(x, t), x, t) - f(t)g(x, t), \quad (2.8)
\end{align*}
\]

where

\[
\begin{align*}
\eta^*(x, \theta, t) &= \int_{\mathbb{R}^n} Z(x - y, t - \theta)\eta(y, \theta)dy, \\
h(x, t) &= V(t)\int_{\mathbb{R}^n} Z(x - y, t)\varphi(y)dy \\
&\quad + \int_0^t \int_{\mathbb{R}^n} Z(x, y, t - s)H(s, t)g(y, s)dy dW(s),
\end{align*}
\]
\[ S(\theta, t) = \int_0^t K(t - \tau)H(\theta, \tau)d\tau. \] (2.9)

**Proof:** Substituting from (2.7) into (2.6) and using Fubini’s theorem, we get the required result.

### 3. The non-exponential stability

Let us consider the stochastic integro partial differential equation:

\[
du(x, t) = \left[ \nabla^2 u(x, t) + F(u(x, t), x, t) \right]dt \\
+ \int_0^t K(t - \theta)u(x, \theta)d\theta dt + f(t)u(x, t)dW(t),
\] (3.1)

with the initial condition

\[ u(x, 0) = \varphi(x). \] (3.2)

In this case we have the following representation of \( u \):

\[
u(x, t) = V(t) \int_{\mathbb{R}^n} Z(x - y, t)\varphi(y)dy \\
+ \int_0^t \int_{\mathbb{R}^n} Z(x - y, t - s)H(s, t)\eta(y, s)dy ds
\] (3.3)

where \( \eta \) is given by:

\[
\eta(x, t) = \int_0^t S(\theta, t)\eta^*(x, \theta, t)d\theta \\
+ \int_0^t K(t - \theta)V(\theta)\int_{\mathbb{R}^n} Z(x - y, \theta)\varphi(y)dy d\theta + G(u_m(x, t), x, t),
\] (3.4)

where \( u \) is given by (3.3).

**Theorem 3.1.** Equation (3.4) has a unique solution \( \eta \) with continuous sample paths on \( \mathbb{R}^n \times [0, T] \) for almost all \( w \in \Omega \).

**Proof.** We notice that \( (\theta, t) \rightarrow S(\theta, t)(\omega) \) and \( (\theta, t) \rightarrow H(\theta, t)(\omega) \) are jointly continuous in both variables for almost all \( w \in \Omega \). Thus equation (3.4) represents a linear Volterra equation with continuous random kernels, a.s., now according to assumption \( A_1 \) and \( A_2 \), we can solve equation (3.4) by the method of successive approximations. Set:

\[
\eta_{m+1}(x, t) = \int_0^t S(\theta, t)\eta_m^*(x, \theta, t)d\theta \\
+ \int_0^t K(t - \theta)V(\theta)\int_{\mathbb{R}^n} Z(x - y, \theta)\varphi(y)dy d\theta + G(u_m(x, t), x, t),
\]

where

\[
u_m(x, t) = V(t) \int_{\mathbb{R}^n} Z(x - y, t)\varphi(y)dy \\
+ \int_0^t \int_{\mathbb{R}^n} Z(x - y, t - s)H(s, t)\eta_m(y, s)dy ds
\]
\( \eta_0(x, t) \) is the zero approximation.

It is easy to see that \( \eta_1, \ldots, \eta_m, \ldots \) have continuous sample paths on \( R^n \times [0, \infty) \) and for almost all \( w \in \Omega \).

Suppressing \( \omega \) dependence for ease expression, it can be seen for almost all \( \omega \in \Omega \):

\[
\sup_{x \in R^n} |\eta_{m+1}(x, t) - \eta_m(x, t)| \leq \frac{(ct)^m}{m!},
\]

for some constant \( c > 0 \) and \( 0 < t \leq T \). Thus the sequence \( \{\eta_m\} \) uniformly converges on \( R^n \times [0, T] \) to \( \eta \) for almost all \( \omega \in \Omega \). It is easy to prove the uniqueness of \( \eta \). Hence we get the required result.

We need the following further assumptions.

\( A_3 \): The kernel \( K \) is strictly positive on \( [0, \infty) \) and has continuous derivative on \( [0, \infty) \).

It is supposed also that \( \int_0^\infty K(t)dt \) exists and

\[
\lim_{t \to \infty} \frac{K'(t)}{K(t)} = 0,
\]

(3.5)

\( A_4 \) : \( \varphi(x) \geq \alpha > 0 \) for all \( x \in R^n \) and \( G(u, x, t) \geq 0 \) for all \( (u, x, t) \in R \times R^n \times [0, \infty) \),

\( A_5 \) : \( \lim_{t \to \infty} \frac{1}{t} \int_0^t f^2(\theta)d\theta = \gamma^2 \) exists,

**Lemma 3.1.** If \( \eta \) is defined by (3.4), then \( \eta(x, t) > 0 \) for all \( (x, t) \in R^n \times (0, \infty) \).

**Proof.** It is easy to see that

\[
\eta(x, t) \geq \int_0^t S(\theta, t)\eta^*(x, \theta, t)d\theta + \alpha S(0, t).
\]

Following Appleby and Reynolds [7], we get the required result, (comp. [8]-[11]).

**Theorem 3.2.** If the unique strong a.s. continuous solution of equation (3.3) satisfies

\[
\lim_{t \to \infty} u(x, t) = 0, \text{ for all } x \in R^n, \text{ a.s.},
\]

then:

(I) \( a + \frac{1}{2}\gamma^2 > 0 \),

(II) \( \lim_{t \to \infty} \inf_{K(t)} \frac{S^*(0, t)}{K(t)} > \xi \), a.s., where \( \xi \) is an almost surely positive random variable,

(III) \( \lim_{t \to \infty} \sup Y(t) = \infty \),

where

\[
Y(t) = \frac{\int_0^t H(\theta, t)K(\theta)d\theta}{K(t)}
\]
(IV) \( \lim_{t \to \infty} \frac{u(x,t)}{K(t)} = \infty \), a.s.,

furthermore:

\[
\lim_{t \to \infty} \sup u(x,t)e^{\epsilon t} = \infty, \text{ a.s.,}
\]

for every \( \epsilon > 0 \) and all \( x \in \mathbb{R}^n \).

**Proof.** Notice that

\[
\lim_{t \to \infty} E\left[ \left\{ \frac{1}{t} \int_0^t f(\theta)dW(\theta) \right\}^2 \right] = \lim_{t \to \infty} \frac{1}{t^2} \int_0^t f^2(\theta)d\theta = 0.
\]

Let \( n^3 \leq t \leq (n+1)^3 \), we get

\[
P\left[ \frac{1}{t} \int_0^t f(\theta)dW(\theta) > \frac{1}{n} \right] \leq \frac{c}{n^2},
\]

for some constant \( c > 0 \). Since \( \sum_{n=1}^\infty \frac{1}{n^2} \) converges, it follows by applying the Borel - Cantelli lemma [12] that

\[
P\left[ \frac{1}{t} \int_0^t f(\theta)dW(\theta) > \frac{1}{n}, \text{i.o.} \right] = 0.
\]

Thus we can deduce that there is a subset \( \Omega^* \) of \( \Omega \) such that \( P(\Omega^*) = 1 \), and for each \( \omega \in \Omega^* \) and for each \( \epsilon > 0 \), there exists \( T(\omega) > 0 \), such that

\[
\int_0^t f(\theta)dW(\theta) > -\epsilon t,
\]

(3.6)

for all \( t > T(\omega) \).

Using \( A_5 \), the inequality (3.6) and lemma 3.1, we get

\[
u(x,t) \geq \alpha V(t) > \alpha \exp[-\epsilon t - (\frac{1}{2} \gamma^2 + a)t].
\]

Now if \( \frac{1}{2} \gamma^2 + a < 0 \), we find

\[
\lim_{t \to \infty} u(x,t) = \infty, \text{ for all } x \in \mathbb{R}^n
\]

Consequently (I) is proved.

To prove (II), let

\[
\beta = \epsilon + a + \frac{1}{2} \gamma^2.
\]

Thus \( V(t) \geq e^{-\beta t}, \beta > 0 \), for all \( t \geq T(\omega) \).

According to [7], one gets

\[
\lim_{t \to \infty} \inf \frac{S(0,t)}{K(t)} = \frac{1}{\beta} e^{-\beta T(\omega)}.
\]
To prove (III), we consider the stochastic process \( \{Y_1(t) : t \geq 0\} \), for which \( Y_1(0) = 1 \) and

\[
dY_1(t) = [1 - (a + \frac{K'(t)Y_1(t)}{K(t)})] \, dt + f(t)Y_1(t)dW(t).
\]

Thus

\[
Y_1(t) = V_1(t)[1 + \int_0^t V_1^{-1}(\theta)d\theta], \quad (3.7)
\]

where

\[
V_1(t) = \exp[\int_0^t f(\theta)dW(\theta) - \int_0^t \left(\frac{1}{2}f^2(\theta) + \frac{K'(\theta)}{K(\theta)} + a\right)d\theta] = \frac{K(0)}{K(t)} V(t).
\]

Clearly

\[
Y_1(t) = Y(t) + \frac{K(0)V(t)}{K(t)}
\]

Using (3.6), one gets

\[
\int_0^t f(\theta)dW(\theta) = \int_0^t [f(\theta) - \gamma]dW(\theta) + \gamma W(t)
\]

\[
\geq - \epsilon t + \gamma W(t), \text{ a.s.,} \quad (3.8)
\]

for sufficiently large \( t \).

Let \( \{Y_2(t) : t \geq 0\} \) be a stochastic process defined by

\[
dY_2(t) = [1 - (a + \epsilon)Y_2(t)]dt + \gamma Y_2(t)dW(t),
\]

\[
Y_2(0) = 1.
\]

Clearly

\[
Y_2(t) = V_2(t)[1 + \int_0^t V_2^{-1}(\theta)d\theta],
\]

where

\[
V_2(t) = \exp[\gamma W(t) - \left(\frac{1}{2}\gamma^2 + a + \epsilon\right)t].
\]

From (3.7) and (3.8), it is easy to get

\[
Y_1(t) \geq Y_2(t). \quad (3.9)
\]

Since \( a + \frac{1}{2}\gamma^2 > 0 \), it follows that

\[
\lim_{t \to \infty} \frac{1}{t} \ln V^{-1}(t) = a + \frac{1}{2}\gamma^2, \text{ a.s.} \quad (3.10)
\]
so using (3.10) together with the assumption $A_3$, one gets
\[ \lim_{t \to \infty} V^{-1}(t)K(t) = \infty, \text{ a.s.} \quad (3.11) \]
Using (3.11) gives
\[ \lim_{t \to \infty} \sup_{t \to \infty} Y(t) = \lim_{t \to \infty} \sup_{t \to \infty} \left[ Y_1(t) - \frac{K(0)}{K(t)}V(t) \right] \]
\[ = \lim_{t \to \infty} \sup_{t \to \infty} Y_1(t) \geq \lim_{t \to \infty} \sup_{t \to \infty} Y_2(t). \]
Let
\[ T_b = \inf \{ t > 0 : Y_2(t) = b \}, \]
\[ P_x \{ T_b < T_\beta \} = \frac{s(x) - s(\beta)}{s(b) - s(\beta)} \]
where $s$ is the scale function of $Y_2(t)$. It is well known that this function is given by
\[ s(x) = e^{2/\gamma^2} \int_1^x y^{21/\gamma^2} e^{2/\gamma^2 y} \, dy, \]
\[ x > 1, \quad c_1 = a + \epsilon. \]
Since $2c_1 + \gamma^2 > 0$, it follows that
\[ s(x) > e^{2/\gamma^2} \int_1^x y^{21/\gamma^2} \, dy, \]
so \[ \lim_{t \to \infty} s(x) = \infty. \]
Thus \[ \lim_{t \to \infty} \sup_{t \to \infty} Y_2(t) = \infty. \]
To prove (IV), we notice that $\eta(x, t) \geq \alpha S(0, t)$, a.s., for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, and this leads to the required result, (comp. [8-11]). I would like to thank Professor Dr. Emil Minchev for his suggestions, and valuable remarks.

**References**


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