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On the Stability of Some Stochastic Integro Partial Differential Equations

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Abstract

Stochastic integro partial differential equations of the form;

$$du(x,t) = \sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial x_i^2}\, dt + F(u(x,t),x,t)dt$$

$$+ \int_0^t K(t-\theta)u(x,\theta)d\theta dt + [f(t)u(x,t) + g(x,t)]dW(t),$$

are considered, where $\{W(t) : t \geq 0\}$ is a standard one-dimensional Wiener process and the kernel K decreases to zero non-exponentially. The behavior of solutions and their convergence to zero are studied. It is proved under suitable conditions that

$$\lim_{t\to\infty} \frac{u(x,t)}{K(t)} = \infty \quad ,$$

almost surely.
The considered stochastic integro partial differential equations arise if we consider the Black-Scholes market consists of a bank account or a bond and a stock. These stochastic models can also applied to population dynamics in biology.

Mathematics Subject Classifications: 34D20, 60H10, 65H20, 45N05

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1. Introduction

In this paper, we consider the stochastic integro partial differential equations of the form

$$du(x,t) \;=\; [\nabla^2 u(x,t) + F(u(x,t),x,t)]dt$$

$$+ \int_0^t K(t-\theta)u(x,\theta)d\theta dt + [f(t)u(x,t) + g(x,t)]dW(t), \quad (1.1)$$

with the initial condition

$$u(x,0) = \varphi(x) \quad (1.2)$$

where φ is a bounded continuous function on the n-dimensional Euclidean space $R^n, x \in R^n, \nabla^2 = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $\{W(t) : t \geq 0\}$ is a standard Wiener process on a complete filtered probability space $\{\Omega, F, F_t, P\}$, where the filtration is the natural one, viz. $F_t = \sigma\{W(s) : 0 \leq s \leq t\}$. The almost sure events considered in this paper are always P-almost sure, (a.s). The functions F, K, f and g satisfy the following conditions:
A_1 : F is continuous on $R \times R^n \times [0,\infty), R = (-\infty,\infty)$ and satisfies the Lipschitz condition;

$$|F(u,x,t) - F(v,x,t)| \leq L|u-v|,$$

for all $(u,x,t),(v,x,t) \in R \times R^n \times [0,\infty)$ and for some positive constant L,
$A_2 : K, f$ are continuous on $[0,\infty)$ and g is continuous on $R^n \times [0,\infty)$.
In section 2, we shall give a suitable representation of the sample paths of the considered problem. The existence and uniqueness of the solutions are treated for more general equations, (see [1-6]). Here we are interested in the stability of solutions. In section 3, we study the phenomenon of the non-exponential stability, (see [7]).
Many special cases of equation (1.1) have several applications to population dynamics in biology and also to Black - Scholes market, which consists of a bank account or a bond and a stock.

2. Representation of solutions

Let $C(R^n \times (0,T])$ be the set of all continuous functions on

$$R^n \times (0,T], T < \infty. \text{ If } u, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x_i^2} \in C(R^n \times (0,T]),$$

$i = 1,...,n$, for almost all $w \in \Omega$ where u is a solution of the Cauchy problem (1.1), (1.2), for almost all $w \in \Omega$, then:

$$\begin{aligned} u(x,t) &= \int_{R^n} Z(x-y,t)\varphi(y)dy \\ &+ \int_0^t \int_{R^n} Z(x-y,t-s)F(u(y,s),y,s)dy\, ds \\ &+ \int_0^t \int_{R^n} Z(x-y,t-s) \int_0^s K(s-\theta)u(y,\theta)d\theta\, dy\, ds \\ &+ \int_0^t \int_{R^n} Z(x-y,t-s)[f(s)u(y,s) + g(y,s)]\, dy\, dW(s), \end{aligned} \quad (2.1)$$

where

$$Z(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{1}{4t}\sum_{i=1}^{n} x_i^2).$$

If $u \in C(R^n \times [0,T])$ satisfies equation (2.1) for almost all $w \in \Omega$, then u is called a mild solution of the Cauchy problem (1.1),(1.2).
If u_1 and u_2 are two solutions, a.s., of (1.1) , (1.2) then assumptions A_1, A_2 and the properties of the Wiener process $W(t)$ lead to the following fact:

$$\begin{aligned}\Lambda(t) &\leq L^2 T \int_0^t \int_{R^n} Z(y,s)\Lambda(s) dy\ ds \\ &+ T^2 \int_0^t \int_0^s \int_{R^n} Z(y,s) K^2(s-\theta)\Lambda(\theta) dy\ d\theta\ ds \\ &+ \int_0^t \int_{R^n} Z(y,s) f^2(s)\Lambda(s) dy\ ds,\end{aligned}$$

where:

$$\Lambda(t) = Sup_{x\in R^n}\ E[\{u_1(x,t) - u_2(x,t)\}^2]$$

and $E(X)$ is the expectation of X. Since $\int_{R^n} Z(y,s)dy = 1$, it follows that

$$\Lambda(t) \leq c\int_0^t \Lambda(s) ds,$$

for some positive constant c. This proves the uniqueness of the mild solutions.
Let us try to find the solution of the considered problem in the form

$$u(x,t) = V(t)\psi(x,t),$$

where $\{V(t) : t \geq 0\}$ is the stochastic process described by the following equation

$$dV(t) = -aV(t) + f(t)\ V(t)dW(t), \tag{2.2}$$

$$V(0) = 1 \tag{2.3}$$

where $a \in R$.
The solution of (2.2), (2.3) is given by

$$V(t) = \exp[\int_0^t f(s)dW(s) - \int_0^t [\frac{1}{2} f^2(s) + a] ds.$$

Let us assume that

$$\{\psi(x,t) : (x,t) \in R^n \times [0,\infty)\}$$

is the stochastic process described by the following equation:

$$d\psi(x,t) = \nabla^2\psi(x,t)dt + A(x,t)dt\ + B(x,t)dW(t), \tag{2.4}$$

$$\psi(x,0)=\varphi(x), \tag{2.5}$$

where A and B are chosen such that $u(x,t)=V(t)\ \psi(x,t)$ satisfies equation (1.1).
We have:

$$\begin{aligned} du(x,t) &= V(t)d\psi(x,t)+\psi(x,t)dV(t) \\ &+ f(t)V(t)B(x,t)dt \end{aligned}$$

Using (2.2) , (2.3), (2.4) and (2.5), we find that it is suitable to choose A and B as follows:

$$A(x,t)=V^{-1}(t)\eta(x,t),$$
$$B(x,t)=V^{-1}(t)g(x,t),$$

where

$$\eta(x,t)=G(u(x,t),x,t)+\int_0^t K(t-\theta)u(x,\theta)d\theta-f(t)g(x,t), \tag{2.6}$$

$$G(u,x,t)=F(u,x,t)+au.$$

Using (2.4), (2.5) and (2.6), one gets the following representation of u;

$$\begin{aligned} u(x,t) &= V(t)\int_{R^n} Z\ (x-y,t)\varphi(y)dy \\ &+ \int_0^t\int_{R^n} Z(x-y,t-s)H(s,t)\eta(y,s)dy\ ds \\ &+ \int_0^t\int_{R^n} Z(x-y,t-s)H(s,t)g(y,s)dy\ dW(s), \qquad (2.7) \end{aligned}$$

where $H(s,t)=V(t)V^{-1}(s)$.
now we prove that the stochastic process $\{\eta(x,t):(x,t)\in R^n\times[0,\infty)\}$ satisfies a nonlinear Riemann integral equation with random kernels.
Lemma 2.1. If the stochastic process $\{\eta(x,t):(x,t)\in R^n\times[0,\infty)\}$ is given by (2.6), then

$$\begin{aligned} \eta(x,t) &= \int_0^t S(\theta,t)\eta^*(x,\theta,t)d\theta \\ &+ \int_0^t K(t-\theta)h(x,\theta)d\theta+G(u(x,t),x,t)-f(t)g(x,t), \qquad (2.8) \end{aligned}$$

where

$$\begin{aligned} \eta^*(x,\theta,t) &= \int_{R^n} Z(x-y,t-\theta)\eta(y,\theta)dy, \\ h(x,t) &= V(t)\int_{R^n} Z(x-y,t)\varphi(y)dy \\ &+ \int_0^t\int_{R^n} Z(x,y,t-s)H(s,t)g(y,s)dy\ W(s), \end{aligned}$$

$$S(\theta,t)=\int_\theta^t K(t-\tau)H(\theta,\tau)d\tau. \tag{2.9}$$

Proof: Substituting from (2.7) into (2.6) and using Fubini's theorem, we get the required result.

3. The non-exponential stability

Let us consider the stochastic integro partial differential equation:

$$\begin{aligned} du(x,t) &= [\nabla^2 u(x,t)+F(u(x,t),x,t)]dt \\ &+ \int_0^t K(t-\theta)u(x,\theta)d\theta dt+f(t)u(x,t)dW(t), \end{aligned} \tag{3.1}$$

with the initial condition

$$u(x,0)=\varphi(x). \tag{3.2}$$

In this case we have the following representation of u:

$$\begin{aligned} u(x,t) &= V(t)\int_{R^n} Z(x-y,t)\varphi(y)dy \\ &+ \int_0^t\int_{R^n} Z(x-y,t-s)H(s,t)\eta(y,s)dy\ ds \end{aligned} \tag{3.3}$$

where η is given by:

$$\begin{aligned} \eta(x,t) &= \int_0^t S(\theta,t)\eta^*(x,\theta,t)d\theta \\ &+ \int_0^t K(t-\theta)V(\theta)\int_{R^n} Z(x-y,\theta)\varphi(y)dy\ d\theta+G(u(x,t),x,t), \end{aligned} \tag{3.4}$$

where u is given by (3.3).

Theorem 3.1. Equation (3.4) has a unique solution η with continuous sample paths on $R^n\times[0,T]$ for almost all $w\in\Omega$.

Proof. We notice that $(\theta,t)\to S(\theta,t)(\omega)$ and $(\theta,t)\to H(\theta,t)(\omega)$ are jointly continuous in both variables for almost all $w\in\Omega$. Thus equation (3.4) represents a linear Volterra equation with continuous random kernels, a.s., now according to assumption A_1 and A_2, we can solve equation (3.4) by the method of successive approximations. Set:

$$\begin{aligned} \eta_{m+1}(x,t) &= \int_0^t S(\theta,t)\eta_m^*(x,\theta,t)d\theta \\ &+ \int_0^t K(t-\theta)V(\theta)\int_{R^n} Z(x-y,\theta)\varphi(y)dy\ d\theta+G(u_m(x,t),x,t), \end{aligned}$$

where

$$\begin{aligned} u_m(x,t) &= V(t)\int_{R^n} Z(x-y,t)\varphi(y)dy \\ &+ \int_0^t\int_{R^n} Z(x-y,t-s)H(s,t)\eta_m(y,s)dy\ ds \end{aligned}$$

$\eta_0(x,t)$ is the zero approximation.
It is easy to see that $\eta_1, ..., \eta_m, ...$ have continuous sample paths on $R^n \times [0,\infty)$ and for almost all $w \in \Omega$.
Suppressing ω dependence for ease expression, it can be seen for almost all $\omega \in \Omega$:

$$\sup_{x \in R^n} |\eta_{m+1}(x,t) - \eta_m(x,t)| \leq \frac{(ct)^m}{m!},$$

for some constant $c > 0$ and $0 < t \leq T$. Thus the sequence $\{\eta_m\}$ uniformly converges on $R^n \times [0,T]$ to η for almost all $\omega \in \Omega$. It is easy to prove the uniqueness of η. Hence we get the required result.
We need the following further assumptions.
A_3**:** The kernel K is strictly positive on $[0,\infty)$ and has continuous derivative on $[0,\infty)$.
It is supposed also that $\int_0^\infty K(t)dt$ exists and

$$\lim_{t\to\infty} \frac{K'(t)}{K(t)} = 0, \tag{3.5}$$

A_4 : $\varphi(x) \geq \alpha > 0$ for all $x \in R^n$ and $G(u,x,t) \geq 0$ for all $(u,x,t) \in R \times R^n \times [0,\infty)$,
A_5**:** $\lim\limits_{t\to\infty} \frac{1}{t} \int_0^t f^2(\theta)d\theta = \gamma^2$ exists,
Lemma 3.1. If η is defined by (3.4), then $\eta(x,t) > 0$ for all $(x,t) \in R^n \times (0,\infty)$.
Proof. It is easy to see that

$$\eta(x,t) \geq \int_0^t S(\theta,t)\eta^*(x,\theta,t)d\theta + \alpha S(0,t).$$

Following Appleby and Reynolds [7], we get the required result, (comp. [8]-[11]).
Theorem 3.2. If the unique strong a.s. continuous solution of equation (3.3) satisfies

$$\lim_{t\to\infty} u(x,t) = 0, \text{for} \quad \text{all } x \in R^n, a.s.,$$

then:
(I) $a + \frac{1}{2}\gamma^2 > 0$,
(II) $\lim\limits_{t\to\infty} \inf \frac{S(0,t)}{K(t)} > \xi$, a.s., where ξ is an almost surely positive random variable,
(III) $\lim\limits_{t\to\infty}$ sup $Y(t) = \infty$,
where

$$Y(t) = \frac{\int_0^t H(\theta,t)K(\theta)d\theta}{K(t)}$$

(IV) $\lim_{t\to\infty} \frac{u(x,t)}{K(t)} = \infty$, a.s.,
furthermore:

$$\lim_{t\to\infty} \ \sup \ u(x,t)e^{\varepsilon t} = \infty, a.s.,$$

for every $\epsilon > 0$ and all $x \in R^n$.

Proof. Notice that

$$\lim_{t\to\infty} E[\{\frac{1}{t}\int_0^t f(\theta)dW(\theta)\}^2] = \lim_{t\to\infty} \frac{1}{t^2}\int_0^t f^2(\theta)d\theta = 0.$$

Let $n^3 \leq t \leq (n+1)^3$, we get

$$P[\frac{1}{t}\int_0^t f(\theta)dW(\theta) > \frac{1}{n}] \leq \frac{c}{n^2},$$

for some constant $c > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows by applying the Borel - Cantelli lemma [12] that

$$P[\frac{1}{t}\int_0^t f(\theta)dW(\theta) > \frac{1}{n}, i.o.] = 0.$$

Thus we can deduce that there is a subset Ω^* of Ω such that $P(\Omega^*) = 1$, and for each $\omega \in \Omega^*$ and for each $\epsilon > 0$, there exists $T(\omega) > 0$, such that

$$\int_0^t f(\theta)dW(\theta) > -\epsilon t, \tag{3.6}$$

for all $t > T(\omega)$.
Using A_5, the inequality (3.6) and lemma 3.1, we get

$$u(x,t) \geq \alpha V(t) > \alpha \exp[-\epsilon t - (\frac{1}{2}\gamma^2 + a)t].$$

Now if $\frac{1}{2}\gamma^2 + a < 0$, we find

$$\lim_{t\to\infty} u(x,t) = \infty, \text{ for all } x \in R^n$$

Consequently (I) is proved.
To prove (II), let

$$\beta = \epsilon + a + \frac{1}{2}\gamma^2.$$

Thus $V(t) \geq e^{-\beta t}, \beta > 0$, for all $t \geq T(\omega)$.
According to [7], one gets

$$\lim_{t\to\infty} \inf \frac{S(0,t)}{K(t)} = \frac{1}{\beta} e^{-\beta T(\omega)}.$$

To prove (III), we consider the stochastic process $\{Y_1(t) : t \geq 0\}$, for which $Y_1(0) = 1$ and

$$dY_1(t) = [1 - (a + \frac{K'(t)Y_1(t)}{K(t))}] \; dt \; + f(t)Y_1(t)dW(t).$$

Thus

$$Y_1(t) = V_1(t)[1 + \int_0^t V_1^{-1}(\theta)d\theta], \tag{3.7}$$

where

$$V_1(t) = \exp[\int_0^t f(\theta)dW(\theta) - \int_0^t \{\frac{1}{2}f^2(\theta) + \frac{K'(\theta)}{K(\theta)} + a\}d\theta]$$

$$= \frac{K(0)}{K(t)} \; V(t).$$

Clearly

$$Y_1(t) = Y(t) + \frac{K(0)V(t)}{K(t)}$$

Using (3.6), one gets

$$\int_0^t f(\theta)dW(\theta) = \int_0^t [f(\theta) - \gamma]dW(\theta) + \gamma W(t)$$

$$\geq \; - \; \epsilon t + \gamma W(t), \; a.s., \tag{3.8}$$

for sufficiently large t.

Let $\{Y_2(t) : t \geq 0\}$ be a stochastic process defined by

$$dY_2(t) = [1 - (a + \epsilon)Y_2(t)]dt + \gamma Y_2(t)dW(t),$$

$$Y_2(0) = 1.$$

Clearly

$$Y_2(t) = V_2(t)[1 + \int_0^t V_2^{-1}(\theta)d\theta],$$

where

$$V_2(t) = \exp[\gamma W(t) - (\frac{1}{2}\gamma^2 + a + \epsilon)t].$$

From (3.7) and (3.8), it is easy to get

$$Y_1(t) \geq Y_2(t). \tag{3.9}$$

Since $a + \frac{1}{2}\gamma^2 > 0$, it follows that

$$\lim_{t \to \infty} \frac{1}{t} \; ln \; V^{-1}(t) = a + \frac{1}{2}\gamma^2, a.s. \tag{3.10}$$

so using (3.10) together with the assumption A_3, one gets

$$\lim_{t\to\infty} V^{-1}(t)K(t) = \infty, \ a.s. \tag{3.11}$$

Using (3.11) gives

$$\begin{aligned}\lim_{t\to\infty}\sup Y(t) &= \lim_{t\to\infty}\sup[Y_1(t) - \frac{K(0)}{K(t)}V(t)] \\ &= \lim_{t\to\infty}\sup Y_1(t) \geq \lim_{t\to\infty}\sup Y_2(t).\end{aligned}$$

Let

$$T_b = \inf\{t > 0 : Y_2(t) = b\},$$

$$P_x\{T_b < T_\beta\} \ = \ \frac{s(x) - s(\beta)}{s(b) - s(\beta)}$$

where s is the scale function of $Y_2(t)$. It is well known that this function is given by

$$s(x) = e^{2/\gamma^2} \int_1^x y^{24/\gamma^2} \ e^{2/\gamma^2 y} \ dy,$$

$$x > 1, \quad c_1 = a \ + \ \epsilon.$$

Since $2c_1 + \gamma^2 > 0$, it follows that

$$s(x) > e^{2/\gamma^2} \int_1^x y^{x_1/\gamma^2} \ dy,$$

so $\lim\limits_{t\to\infty} s(x) = \infty$.
Thus $\lim\limits_{t\to\infty} \sup \ Y_2(t) = \infty$.
To prove (IV), we notice that $\eta(x,t) \geq \alpha S(0,t)$, a.s., for all $(x,t) \in R^n \times [0,\infty)$, and this leads to the required result, (comp. [8-11]). I would like to thank Professor Dr. Emil Minchev for his suggestions, and valuable remarks.

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