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Approximate Solutions for Nonlinear Fractional Heat Equations

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Abstract

In this paper, we shall study the approximate solutions of the Cauchy problem,

$$(D_{0^+}^{\alpha}u)(x,t)=\sum_{i=1}^{n}\frac{\partial^2(x,t)}{\partial x_i^2}+f(x,t,u(x,t)),$$
$$u(x,0)\ =\ \varphi(x),$$

where $0<\alpha<1$, and $f(x,t,u)$ is a continuous function of the three variables x,t and u on the domain

$$Q=\{(x,t,u):\ x\in R^n, 0\le t\le h, |u-\varphi|\le\mu, \mu>0\}.$$

Also, some properties of the solutions of the considered problem are established.

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1. Introduction:

In this paper we consider the approximate solutions of the problem

$$(D_{0^+}^{\alpha}u)(x,t)\ =\ \sum_{i=1}^{n}\frac{\partial^2u(x,t)}{\partial x_i^2}+f(x,t,u(x,t)),\tag{1.1}$$

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$$u(x,0) = \varphi(x), \tag{1.2}$$

with $0 < \alpha < 1$ over the interval [0,a], [1,2].
Suppose that $f(x,t,u)$ is continuous on the domain Q.
Thus

$$\max_{(x,t,u)\in Q} \ |f(x,t,u)| \ = \ M < \infty. \tag{1.3}$$

According to [3], the Cauchy problems (1.1), (1.2) are equivalent to the integral equation

$$u(x,t) = \int_0^\infty \int_{R_n} \zeta_\alpha(\theta) G(x-\xi, t^\alpha\theta)\varphi(\xi) d\xi \ d\theta$$

$$+\alpha \int_0^t \int_0^\infty \int_{R_n} \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta) G(x-\xi,(t-\eta)^\alpha\theta) f(\xi,\eta,u(\xi,\eta)) d\xi d\theta d\eta. \tag{1.4}$$

where $\zeta_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$ and

$$G(x,t) = \frac{e^{-|x|^2/4t}}{(\sqrt{4\pi t})^n}, \qquad |x|^2 = x_1^2 + x_2^2 + ... + x_n^2, [4,5].$$

2. Approximate solutions:

To construct the approximate solutions of (1.4) let us choose $u_1(x,t)$ by[6]

$$u_1(x,t) = \int_0^\infty \int_{R^n} \zeta_\alpha(\theta) G(x-\xi, t^\alpha\theta)\varphi(\xi) d\xi d\theta$$

$$+\alpha \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) G(x-\xi,(t-\eta)^\alpha\theta) f(\xi,\eta,\alpha_1(\xi)) d\xi d\theta d\eta. \tag{1.5}$$

and

$$\alpha_1(x) = \frac{1}{h}\int_0^h u_1(x,t)dt. \tag{1.6}$$

From (1.5) and (1.6) we get

$$h\alpha_1(x) = \int_0^h \int_0^\infty \int_{R^n} \zeta_\alpha(\theta) G(x-\xi, t^\alpha\theta)\varphi(\xi) d\theta dt$$

$$+\alpha \int_0^h \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) G(x-\xi,(t-\eta)^\alpha\theta) f(\xi,\eta,\alpha_1(\xi)) d\xi d\theta d\eta. \tag{1.7}$$

For the second approximation let us write

$$u_2(x,t) = \int_0^\infty \int_{R^n} \zeta_\alpha(\theta) G(x-\xi, t^\alpha\theta)\varphi(\xi) d\xi d\theta$$

$$+\alpha \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) G(x-\xi,(t-\eta)^\alpha\theta) f(\xi,\eta,u_1+\alpha_2) d\xi d\theta d\eta, \tag{1.8}$$

$$\alpha_2(x) = \frac{1}{h}\int_0^h \ \delta_2(x,t)dt,\ \ \delta_2(x,t) = u_2(x,t) - u_1(x,t). \tag{1.9}$$

According to (1.5) and (1.8) we get

$$\delta_2(x,t) = \alpha\int_0^t\int_0^\infty\int_{R^n} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)[f(\xi,\eta,u_1+\alpha_2)$$

$$-f(\xi,\eta,\alpha_1)]d\xi d\theta d\eta, \tag{1.10}$$

and so, according to formula (1.9) we have the following equation

$$h\alpha_2(x) = \alpha\int_0^h\int_0^t\int_0^\infty\int_{R^n} \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)[f(\xi,\eta,u_1+\alpha_2)$$

$$-f(\xi,\eta,\alpha_1)]d\xi d\theta d\eta dt. \tag{1.11}$$

continuing this process m times, we get

$$u_m(x,t) = \int_0^\infty\int_{R^n}\zeta_\alpha(\theta)G(x-\xi,t^\alpha\theta)\varphi(\xi)d\xi d\theta$$

$$+\alpha\int_0^t\int_0^\infty\int_{R^n}\theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)f(\xi,\eta,u_{m-1}+\alpha_m)d\xi d\theta d\eta$$

$$(m = 3, 4, ...), \tag{1.12}$$

where

$$\alpha_m(x) = \frac{1}{h}\int_0^h \delta_m(x,t)dt, \delta_m(x,t) = u_m(x,t) - u_{m-1}(x,1). \tag{1.13}$$

and so from (1.13)

$$\delta_m(x,t) = \alpha\int_0^t\int_0^\infty\int_{R^n}\theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)[f(\xi,\eta,v_m)$$

$$-f(\xi,\eta,v_{m-1}]d\xi d\theta d\eta, \tag{1.14}$$

where

$$v_m = u_{m-1} + \alpha_m.$$

Then $\alpha_m(x)$ will be defined by the equation

$$h\alpha_m(x) = \alpha\int_0^h\int_0^t\int_0^\infty\int_{R^n}\theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)[f(\xi,\eta,v_m)$$

$$-f(\xi,\eta,v_{m-1}]d\xi d\theta d\eta dt. \tag{1.15}$$

Suppose that $f(x,t,u))$ satisfies the Lipschitz condition with respect to u :

$$|f(x,t,u) - f(x,t,v)| \leq L|u-v|, \quad L > 0. \tag{1.16}$$

Theorem (1.1):

Suppose $0 < \alpha < 1, f(x,t,u(x,t))$ is continuous function on domain

$$Q = \{(x,t,u) : x \in R^n,\ 0 \leq t \leq h, |u-\varphi| \leq \mu, \mu > 0\},$$

and suppose

$$\frac{Lh^\alpha}{\alpha+1} \ < \ 1, \tag{1.17}$$

$$Mh^\alpha + 2\lambda + \frac{2Mh^\alpha}{\alpha+1} \ \leq \mu,\ |\varphi(x)| \leq \lambda. \tag{1.18}$$

Then equation (1.7) has a unique real root $\alpha_1(x)$ satisfies the condition

$$|\alpha_1(x) - \varphi(x)| \ < \ \mu. \tag{1.19}$$

In addition equation (1.15) has a unique real root $\alpha_m, m = 2, 3, ...$ satisfies the condition

$$|\alpha_m(x)| \leq \frac{2Mh^\alpha}{\alpha+1}. \tag{1.20}$$

and so

$$|u_{m-1}(x,t) + \alpha_m(x) - \phi(x)| \ \leq \ \mu.$$

proof:

Let us choose the notations

$$\theta(c(x)) = c(x) - \frac{1}{h}\int_0^h \int_0^\infty \int_{R^n} \zeta_\alpha(\theta) G(x-\xi, t^\alpha\theta)\varphi(\xi) d\xi d\theta dt$$

$$-\frac{\alpha}{h}\int_0^h \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) G(x-\xi, (t-\eta)^\alpha\theta) f(\xi,\eta,c) d\xi d\theta d\eta dt,$$

$$\psi_m(c(x)) = c(x)$$

$$-\frac{\alpha}{h}\int_0^h \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) G(x-\xi, (t-\eta)^\alpha\theta)[f(\xi,\eta,u_{m-1}+c)$$

$$-f(\xi,\eta,u_{m-2}+\alpha_{m-1})] d\xi d\theta d\eta dt$$

$$m = 2, 3, ...$$

From (1.3) and (1.18), we get

$$\theta(-\mu + \varphi(x)) \ \leq \ \ -\mu + \varphi + \lambda + \frac{Mh^\alpha}{\alpha+1} \ \leq \ 0\ ,$$

$$\theta(\mu + \varphi(x)) \ \geq \ \mu + \varphi - \lambda - \frac{Mh^\alpha}{\alpha+1} \ \geq \ 0\ .$$

This means that the equation $\theta(c) = 0$ has root $\alpha_1(x)$ satisfies the condition (1.19).We will see that on the interval $[-\mu+\varphi(x), \mu+\varphi(x)]$ this root is unique. Suppose that there is another root $\alpha_1^{`}(x)$, then

$$|\alpha_1^{`} - \alpha_1| = \frac{\alpha}{h}|\int_0^h \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)[f(\xi,\eta,\alpha_1^1)$$

$$-f(\xi,\eta,\alpha_1)]d\xi d\theta d\eta dt|$$

$$\leq \frac{Lh^\alpha}{\alpha+1}\,|\alpha_1^{`} - \alpha_1|$$

and so $\frac{Lh^\alpha}{\alpha+1} \geq 1$ which contradicts the inequality (1.17).
According to (1.5) and (1.3), $|u_1(x,t) - \varphi(x)| \leq 2\lambda + Mh^\alpha$
then

$$\varphi(x) - 2\lambda - Mh^\alpha \leq u_1(x,t) \leq \varphi(x) + Mh^\alpha + 2\lambda \tag{1.21}$$

From inequality (1.18)

$$Mh^\alpha \leq \mu - 2\lambda - \frac{2Mh^\alpha}{\alpha+1}$$

From (1.21) and (1.22) we get

$$\varphi(x) - \mu + \frac{2Mh^\alpha}{\alpha+1} \leq u_1(x,t) \leq \varphi(x) + \mu - \frac{2Mh^\alpha}{\alpha+1},$$

$$\varphi(x) - \mu \leq u_1(x,t) + |\frac{2Mh^\alpha}{\alpha+1}| \;\leq\; \varphi(x) + \mu.$$

In addition

$$\psi_2(\frac{-2Mh^\alpha}{\alpha+1}) = \frac{-2Mh^\alpha}{\alpha+1}$$

$$-\frac{\alpha}{h}\int_0^h \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)[f(\xi,\eta,u_2-\frac{2Mh^\alpha}{\alpha+1})$$

$$-f(\xi,\eta,\alpha_1)]d\xi d\theta d\eta dt$$

$$\leq \frac{-2Mh^\alpha}{\alpha+1} - \frac{2Mh^\alpha}{\alpha+1} \leq 0.$$

Similarly

$$\psi_2(\frac{2Mh^\alpha}{\alpha+1}) \geq 0 \text{ and so equation}$$

$$\psi_2(c) \;=\; 0 \text{ has a root } \alpha_2(x) \in R$$

satisfies the condition

$$|\alpha_2(x)| \leq \frac{2Mh^\alpha}{\alpha+1}$$

And so $|u_1(x,t)+\alpha_2(x)-\varphi(x)| \leq M$
In a similar manner, it is easy to see that on the interval $[\frac{-2Mh^\alpha}{\alpha+1}, \frac{2Mh^\alpha}{\alpha+1}]$, the root $\alpha_2(x)$ is unique
In general

$$|u_{m-2}(x,t)+\alpha_{m-1}(x)-\varphi(x)| \leq M \qquad (m=3,4,.....),$$

$$|u_{m-1}(x,t)-\varphi(x)| \leq \frac{Mh^\alpha}{\alpha+1},$$

$$-\mu+\varphi(x) \leq u_{m-1}(x,t)+|\frac{2Mh^\alpha}{\alpha+1}| \leq \mu+\varphi(x)$$

and so there exist a root $\alpha_m(x) \in R$ of the equation $\psi_m(c)=0$ which satisfies the condition

$$|\alpha_m(x)| \leq \frac{2Mh^\alpha}{\alpha+1}, \text{and so}$$

$$|u_{m-1}(x,t)+\alpha_m(x)-\varphi(x)| \leq \mu.$$

Hence the required result.
Suppose on the interval [0,h]

$$|u_1(x,t)-\varphi(x)| \leq \delta,\ (\delta>0). \tag{1.23}$$

$$\varepsilon = \frac{Lh^\alpha(\alpha+2)}{(\alpha+1)-Lh^\alpha}. \tag{1.24}$$

Theorem 1.2:
Suppose $0<\alpha<1, f(x,t,u(x,t))$ satisfies the condition of theorem (1.1), the inequalities (1.17) and (1.24) are satisfies, then the sequence $\{u_m(x,t)\}$ which constructed by the previous method uniformly converges on the interval [0,h] to a function u(x,t) which is the solution of the equation (1.4).
proof:
From (1.11), (1.16) and (1.23) we get

$$|\alpha_2(x)|$$

$$\leq \frac{\alpha L}{h}\int_0^h\int_0^t\int_0^\infty\int_{R^n}\theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)|u_1(x,t)+\alpha_2(x)-\alpha_1(x)|d\xi d\theta d\eta dt,$$

$$\leq \frac{\alpha L|\alpha_2(x)|}{h}\int_0^h\int_0^t\int_0^\infty\int_{R^n}\theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)\ d\xi\ d\theta\ d\eta\ dt.$$

$$+\frac{\alpha L\delta}{h}\int_0^h\int_0^t\int_0^\infty\int_{R^n}\theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)G(x-\xi,(t-\eta)^\alpha\theta)d\xi\ d\theta\ d\eta\ dt,$$

$$\leq \frac{Lh^\alpha|\alpha_1(x)|}{\alpha+1} + \frac{L\delta h^\alpha}{\alpha+1} = \frac{L\delta h^\alpha}{(\alpha+1)-Lh^\alpha} \equiv G_1,$$

Then

$$|\alpha_2(x)| \leq \ G_1 \equiv \frac{L\delta h^\alpha}{(\alpha+1)-Lh^\alpha} = \delta\frac{\varepsilon - Lh^\alpha}{1+Lh^\alpha} \equiv G_\varepsilon \tag{1.25}$$

Thus from (1.10) and using (1.16), (1.23) and (1.25) we get

$$\begin{aligned} |\delta_2(x,t)| \ &\leq \ L\alpha|u_1(x,t)+\alpha_2(x)-\alpha_1(x)| \int_0^t (t-\eta)^{\alpha-1}\, d\eta, \\ &\leq \ Lh^\alpha\{\delta + |\alpha_2(x)|\} \ = \ Lh^\alpha\delta\{1+\frac{Lh^\alpha}{(\alpha+1)-Lh^\alpha}\}, \end{aligned}$$

$$|\delta_2(x,t)| \leq \frac{Lh^\alpha\delta(\alpha+1)}{(\alpha+1)-Lh^\alpha} \equiv G_2,$$

and so

$$|\delta_2(x,t)| + |\alpha_2(x)| \leq \frac{Lh^\alpha\delta(\alpha+1)}{(\alpha+1)-Lh^\alpha} \ + \ \frac{Lh^\alpha\delta}{(\alpha+1)-Lh^\alpha} = \frac{Lh^\alpha\delta(\alpha+2)}{(\alpha+1)-Lh^\alpha} = \delta\varepsilon \tag{1.26}$$

From (1.15) at m = 3 and (1.16) we get

$$|\alpha_3(x)| \leq \frac{\alpha L}{h}\int_0^h \ \int_0^t (t-\eta)^{\alpha-1} \ |\delta_2(x,t)+\alpha_3(x)-\alpha_2(x)| \ d\eta \ dt$$

From the inequality (1.26), we get the estimation

$$|\alpha_3(x)| \ \leq \ \frac{Lh^\alpha\delta}{(\alpha+1)-Lh^\alpha}\ \varepsilon = G_1\varepsilon.$$

Then from (1.14) at n = 3,

$$|\delta_3(x,t)|$$

$$\leq \alpha L \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) G(x-\xi,(t-\eta)^\alpha\theta) \ |u_2(x,t)+\alpha_3-u_1-\alpha_2| d\xi \ d\theta \ d\eta$$

$$\leq Lh^\alpha\{\delta\varepsilon + \frac{Lh^\alpha\delta\varepsilon}{(\alpha+1)-Lh^\alpha}\} = \frac{Lh^\alpha\delta\varepsilon(\alpha+1)}{(\alpha+1)-Lh^\alpha} = G_2\varepsilon$$

In general, if $|\alpha_{m-1}(x)| \leq G_1\varepsilon^{m-3}$,

$$|\delta_{m-1}(x,t)| \leq G_2\varepsilon^{m-3}, \text{ then}$$

$$|\delta_{m-1}(x,t)| + |\alpha_{m-1}(x)| \leq \delta\varepsilon^{m-2},$$

and from (1.15), we get

$$|\alpha_m(x)| \leq \frac{\alpha L}{h}\int_0^h \int_0^t (t-\eta)^{\alpha-1} \ |\delta_{m-1}(x,t)+\alpha_m(x)-\alpha_{m-1}(x)| \ d\eta \ dt$$

Then $|\alpha_m(x)| \leq G_1 \varepsilon^{m-2}$, and so from formula (1.14) we get $|\delta_m(x)| \leq G_2 \epsilon^{m-2}$ Consequently, from the inequality (1.24), when $m \to \infty, u_m(x,t)$ on the interval [0,a]uniformly converges to $u(x,t)$, which satisfy (1.4).

Theorem 1.3:

Suppose $0 < \alpha < 1, f(x,t,u(x,t))$ satisfies the conditions of theorem (1.2) and also (1.17) is satisfied. Then $u_m(x,t)$ which constructed by the previous method is approximated by $u(x,t)$ on the interval [0,a] and so the following inequality is satisfied

$$|u(x,t) - u_m(x,t)| \leq \frac{\delta\eta}{1 - Lh^\alpha} \{1 - \frac{2Lh^\alpha}{1 + Lh^\alpha}(1 - \eta^{m-1})\}\varepsilon^m,$$

$$\eta \;=\; (\alpha + 1) - \frac{Lh^\alpha}{\alpha + 2}$$

proof:

Using (1.4), (1.5) and the inequality (1.16), we get

$$|u - u_1|$$

$$= \alpha \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1} \zeta_\alpha(\theta) G(x-\xi,(t-\eta)^\alpha\theta) \, |f(\xi,\eta,u(\xi,\eta)) - f(\xi,\eta,\alpha_1(\xi))| \, d\xi \, d\theta \, d\eta,$$

$$\begin{aligned} &\leq \;\; \alpha L \int_0^t (t-\eta)^{\alpha-1}\{|u - u_1| \;+\; |u_1 - \alpha_1|\} \; dt, \\ &\leq \;\; \frac{\delta Lh^\alpha}{1 - Lh^\alpha} \equiv G_3. \end{aligned}$$

From (1.1), (1.8) and the inequality (1.16) we can deduce

$$|u - u_2|$$

$$= \alpha \int_0^t \int_0^\infty \int_{R^n} \theta(t-\eta)^{\alpha-1} \xi_\alpha(\theta) G(x-\xi,(t-\eta)^\alpha\theta) |f(\xi,\eta,u(\xi,\eta)) - f(\xi,\eta,u_1(\xi,\eta) + \alpha_2)| d\xi d\theta d\eta$$

$$\leq \;\; \alpha L \int_0^t (t-\eta)^{\alpha-1}\{|u - u_1| + |\alpha_2|\} d\eta.$$

and then from (1.25) and (1.27)

$$|u - u_2| \leq (G_3 + G_\varepsilon)Lh^\alpha = \frac{\delta Lh^\alpha}{1 + Lh^\alpha} \; (1 + \frac{2Lh^\alpha}{1 - Lh^\alpha}\eta)\varepsilon,$$

$$\eta \;=\; \frac{Lh^\alpha}{\varepsilon} \;=\; \frac{(\alpha + 1) - Lh^\alpha}{\alpha + 2}$$

According to (1.1), formula (1.12) and the inequality (1.6) we can write

$$\begin{aligned} |u-u_m| &\leq \alpha L \int_0^t (t-\eta)^{\alpha-1} \, |u-u_{m-1}-\alpha_m| \, d\eta \\ &\leq Lh^\alpha(|u-u_{m-1}|+|\alpha_m|). \end{aligned}$$

From (1.20) and the inequality

$$|u-u_{m-1}| \leq \frac{\delta L h^\alpha}{1+Lh^\alpha} \quad \left(1+\frac{2Lh^\alpha}{1-Lh^\alpha}\eta^{m-2}\right)\varepsilon.$$

we get

$$\begin{aligned} |u-u_m| &\leq \frac{\delta L h^\alpha}{1+Lh^\alpha} \quad \left(1+\frac{2Lh^\alpha}{1-Lh^\alpha}\eta^{m-1}\right)\varepsilon^{m-1} \\ &= \frac{\delta\eta}{1-Lh^\alpha}\left(1-\frac{2Lh^\alpha}{1+Lh^\alpha}(1-\eta^{m-1})\right)\varepsilon^m, (see[7]). \end{aligned}$$

Hence the required result.

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