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Zero-Divisor Graphs of Idealizations with Respect to Prime Modules

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Abstract

Let R be a commutative ring with identity and let M be a prime R-module. Let $R(+)M$ be the idealization of the ring R by the R-module M. We study the diameter and girth of the zero-divisor graph of the ring $R(+)M$.

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1 Introduction

Throughout All rings are assumed to be commutative rings with non-zero identity. The zero-divisor graph of a ring is the (simple) graph whose vertex set is the set of non-zero zero-divisors, and an edge is drawn between two distinct vertices if their product is zero. This definition is introduced by D. F. Anddeson P. S. Livingston in [1]. In [3] Beck introduced the concept of a zero-divisor graph of a commutative ring. However, he lets all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. In recent years, the study of zero-divisor graph has grown in various directions. At the heart is the interplay between the ring theoretic properties of a ring and the graph theoretic properties of its zero-divisor graph. The zero-divisor graph of a commutative ring has been studied extensively by several authors, e.g. [1, 2, 4]. Our aim in this note is to study the diameter and girth of the zero-divisor graph of the ring $R(+)M$, where M is a prime R-module.

Let R be a commutative ring with non-zero identity. We use the notation A^* to refer to the non-zero elements of A. For two distinct vertices a and b in a graph $\Gamma(R)$, the distance between a and b, denoted $\mathrm{d}(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $\mathrm{d}(a,b)=\infty$. The diameter of a graph $\Gamma(R)$ is $\mathrm{diam}(\Gamma)=sup\{d(a,b):$ a and b are distinct vertices of $\Gamma\}$. We will use use the notation $\mathrm{diam}(\Gamma(R))$ to denote the diameter of the graph of $Z^*(R)$. A graph is said to be connected if there exists a path between any two distinct vertices, and a graph is complete if it is connected with diameter one. The girth of a graph Γ, denoted by $g(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise, $g(\Gamma)=\infty$. We will use the notation $g(\Gamma(R))$ to denote the girth of the graph $Z^*(R)$.

Let M be an R-module. Consider $R(+)M=\{(a,m):a\in R, m\in M\}$ and let (a,m) and (b,n) be two elements of $R(+)M$. Define: $(a,m)+(b,n)=(a+b,m+n)$ and $(a,m)(b,n)=(ab,am+bn)$. Under this definitions $R(+)M$ becomes a commutative ring with identity. Call this ring the idealization of M in R [5]. Let M be an R-module. Then M is called prime if whenever $rm=0$ either $m=0$ or $rM=0$. So M is prime if and only if for every non-zero submodule N of M we have $(0:_R N)=(0:_R M)$. In this case, $P=(0:_R M)$ is a prime ideal of R and we also say that M is a P-prime R-module.

2 Diameter and Girth of $\Gamma(R(+)M)$

Let M be a P-prime module over a Commutative ring R. The notation below will be kept in this paper: $V_1=\{(0,m):m\in M^*\}$, $V_2=\{(a,n):a\in P^*,n\in M\}$ and $V_3=\{(a,n):a\in Z^*(R),n\in M\}$.

Proposition 2.1 *Let R be a commutative ring and let M be a P-prime R-module. Then:*

(i) If $P\neq 0$, then $(a,m)\in Z(R(+)M)$ if and only if $a\in P\cup Z(R)$.

(ii) If $P=0$, then $(a,m)\in Z(R(+)M)$ if and only if $a=0$ and $m\in M^*$.

Proof. (i) Let $(a,m)\in Z(R(+)M))$. We may assume that $a\neq 0$. There exist a non-zero element (b,n) of $R(+)M$ such that $(a,m)(b,n)=(ab,an+bm)=(0,0)$. If $b=0$, then $a\in(0:_R n)=P$; if $b\neq 0$, then $a\in Z(R)$. Conversely, assume that $(a,m)\in R(+)M$ with $a\in P\cup Z(R)$. If $a\in Z(R)$, then $ab=0$ for some non-zero element $b\in R$. If $b\in P$, then $(a,m)(b,0)=(0,0)$. If $b\notin P$, then there is an element x of M such that $bx\neq 0$. Then $(a,m)(0,bx)=(0,0)$. Finally, if $a\in P$, then there exists a non-zero element y of M such that $ay=0$. Therefore, $(a,m)(0,y)=(0,0)$, and so the proof is complete.

(ii) Let $(a, m) \in Z(R(+)M))$. We may assume that $a \neq 0$. There exist a non-zero element (b, n) of $R(+)M$ such that $ab = 0$ and $an + bm = 0$. Since M is a 0-prime R-module, we must have R is an integral domain; hence if $a \neq 0$, then $b = 0$, $n \neq 0$ and $a \in (0 :_R n) = 0$ which is a contradiction. Therefore, $a = 0$ and $m \neq 0$ since $(a, m) \neq 0$. The other implication is clear. □

Theorem 2.2 *Let R be a commutative ring and let M be a P-prime R-module. Then:*

(i) If $P = 0$, then $Z(R(+)M)^* = V_1$.

(ii) If $P \neq 0$ and $Z(R)^* \neq \emptyset$, then $Z(R(+)M)^* = V_1 \cup V_2 \cup V_3$.

(iii) If $P \neq 0$ and $Z(R)^* = \emptyset$, then $Z(R(+)M)^* = V_1 \cup V_2$.

Proof. This follows from Proposition 2.1. □

Theorem 2.3 *Let M be a prime module over a commutative ring R and let $\Gamma(R) \neq \emptyset$. Then $\Gamma(R(+)M)$ is complete if and only if $Z(R) \subseteq (0 :_R M)$.*

Proof. Since $\Gamma(R) \neq \emptyset$, we must have $(0 :_R M) = P \neq 0$. Assume $\Gamma(R(+)M)$ is complete. Let $r \in Z(R)$, $0 \neq m \in M$. We may assume thar $r \neq 0$. Then Theorem 2.2 gives $(0, m), (r, 0) \in Z(R(+)M)^*$; hence $(r, 0)(0, m) = (0, 0)$. Therefore, $r \in (0 :_R m) = P$. Conversely, assume that $Z(R) \subseteq P$ and let $(a, m), (b, n) \in Z(R(+)M)^*$. If $a = b = 0$, then clearly $(a, m)(b, n) = (0, 0)$. If $b = 0$ and $a \in Z^*(R) \subseteq P$, then $an = 0$; hence $(a, m)(b, n) = (0, 0)$. If $a, b \in Z^*(R) \subseteq P$, then $an = 0 = bm$, so $(a, m)(b, n) = (0, 0)$. Thus $\Gamma(R(+)M)$ is complete. □

Note that if M is a prime R-module, then any non-zero submodule of M is prime. Therefore, by Theorem 2.3, we have the following corollary:

Corollary 2.4 *Let R be a commutative ring, M a prime R-module, N a non-zero submodule of M and $\Gamma(R) \neq \emptyset$. Then $\Gamma(R(+)M)$ is complete if and only if $\Gamma(R(+)N)$ is complete.*

Proposition 2.5 *Let M be a P-prime module over a commutative ring R and let $\Gamma(R) = \emptyset$. Then:*

(i) If $P = 0$, then $\Gamma(R(+)M)$ is complete.

(ii) If $P \neq 0$, then $\operatorname{diam}(\Gamma(R(+)M)) = 2$.

Proof. (i) Since $\Gamma(R) = \emptyset$, we must have R is an integral domain. If $P = 0$, then Theorem 2.2 gives $Z(R(+)M)^* = V_1$, so clearly it is complete.

(ii) If $P \neq 0$, then Theorem 2.2 gives $Z(R(+)M)^* = V_1 \cup V_2$. Let $z_1 = (a, m), z_2 = (b, n) \in Z(R(+)M)^*$. If $z_1, z_2 \in V_1$, then $z_1 z_2 = 0$. If $z_1 \in V_2$ and $z_2 \in V_1$, then $a \in P$ and $b = 0$; hence $z_1 z_2 = 0$. Similarly, if $z_1 \in V_1$ and $z_2 \in V_2$, then $z_1 z_2 = 0$. So suppose that $z_1, z_2 \in V_2$ and let $0 \neq x \in M$. Then $a, b \in P$; hence $z_1 - (0, x) - z_2$ is a path, as required. □

Theorem 2.6 *Let M be a P-prime module over a commutative ring R and let $\Gamma(R) = \emptyset$. Then* $\text{diam}(\Gamma(R(+)M)) \leq 2$.

Proof. This follows from Proposition 2.5. □

Example 2.7 *(i) Since Z is a 0-prime Z-module, we must have $\Gamma(R(+)M)$ is complete by Theorem 2.5 (i).*

(ii) Let $M = Z_3$ denote the ring of integers modulo 3. Then M is a $3Z$-prime Z-module. Then $\text{diam}(\Gamma(R(+)M)) = 2$ *by Theorem 2.5 (ii).*

Let R be a commutative ring and let M be a P-prime R-module. If $|M| \geq 4$, then $g(\Gamma(R(+)M)) = 4$ by [2, p. 237]. Then we only need to consider when the P-prime module M has two or three elements. Also, since the module M is unitary, the ring R cannot have fewer than three elements. So throughout this section we shall assume unless otherwise stated, that $|R| \geq 3$.

Lemma 2.8 *Let R be a commutative ring with identity and $M \cong Z_3$ a P-prime R-module. Then:*

(i) $P \neq 0$ if and only if $|R| > 3$.

(ii) $P = 0$ if and only if $|R| = 3$.

Proof. (i) Since $P \neq 0$ and it is prime, we must have $|P| \geq 3$; hence $|R| \geq 4$. Conversely, assume that $|R| \geq 4$, so by [2, p. 237], there always exists a non-zero $r \in R$ such that $rZ_3 = 0$. Therefore, $P \neq 0$. (ii) Is clear. □

Theorem 2.9 *Let R be a commutative ring with identity and $M \cong Z_3$ a P-prime R-module. Then*

(i) $g(\Gamma(R(+)M)) = 3$ if and only if $|R| > 3$

(ii) $g(\Gamma(R(+)M)) = \infty$ if and only if $|R| = 3$

Proof. This follows from Lemma 2.8 and [2, Theorem 2.1]. □

Corollary 2.10 *Let R be a commutative ring with identity and $M \cong Z_2$ a P-prime R-module. Then:*

(i) If $P = 0$, then $g(\Gamma(R(+)M)) = \infty$.

(i) If $P \neq 0$ and $\Gamma(R) = \emptyset$, then $g(\Gamma(R(+)M)) = \infty$.

(i) If $Z^*(R) \subseteq P \neq 0$, $g(\Gamma)(R) = 3$ and $a^2 = 0$ for some $a \in P$, then $g(\Gamma(R(+)M)) = 3$.

Proof. This follows from [2, Theorem 2.2] and Proposition 2.1. □

Example 2.11 *(i) Let $M = Z_3$ denote the ring of integers modulo 3. Then M is a $3Z$-prime Z-module. Then* $\text{g}(\Gamma(R(+)M)) = 3$ *by Theorem 2.9 (i).*

(ii) Let $M = Z_2$ denote the ring of integers modulo 2. Then M is a $2Z$-prime Z-module. Then $\text{g}(\Gamma(R(+)M)) = \infty$ *by Corollary 2.10 (ii).*

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