Zero-Divisor Graphs of Idealizations with Respect to Prime Modules

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Abstract

Let $R$ be a commutative ring with identity and let $M$ be a prime $R$-module. Let $R(+)M$ be the idealization of the ring $R$ by the $R$-module $M$. We study the diameter and girth of the zero-divisor graph of the ring $R(+)M$.

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1 Introduction

Throughout all rings are assumed to be commutative rings with non-zero identity. The zero-divisor graph of a ring is the (simple) graph whose vertex set is the set of non-zero zero-divisors, and an edge is drawn between two distinct vertices if their product is zero. This definition is introduced by D. F. Anddeson P. S. Livingston in [1]. In [3] Beck introduced the concept of a zero-divisor graph of a commutative ring. However, he lets all elements of $R$ be vertices of the graph and his work was mostly concerned with coloring of rings. In recent years, the study of zero-divisor graph has grown in various directions. At the heart is the interplay between the ring theoretic properties of a ring and the graph theoretic properties of its zero-divisor graph. The zero-divisor graph of a commutative ring has been studied extensively by several authors,
e.g. [1, 2, 4]. Our aim in this note is to study the diameter and girth of the zero-divisor graph of the ring $R(+)M$, where $M$ is a prime $R$-module.

Let $R$ be a commutative ring with non-zero identity. We use the notation $A^*$ to refer to the non-zero elements of $A$. For two distinct vertices $a$ and $b$ in a graph $\Gamma(R)$, the distance between $a$ and $b$, denoted $d(a, b)$, is the length of the shortest path connecting $a$ and $b$, if such a path exists; otherwise, $d(a, b) = \infty$. The diameter of a graph $\Gamma(R)$ is $\text{diam}(\Gamma) = \sup\{d(a, b) : a$ and $b$ are distinct vertices of $\Gamma\}$. We will use the notation $\text{diam}(\Gamma(R))$ to denote the diameter of the graph of $Z^*(R)$. A graph is said to be connected if there exists a path between any two distinct vertices, and a graph is complete if it is connected with diameter one. The girth of a graph $\Gamma$, denoted by $g(\Gamma)$, is the length of a shortest cycle in $\Gamma$, provided $\Gamma$ contains a cycle; otherwise, $g(\Gamma) = \infty$. We will use the notation $g(\Gamma(R))$ to denote the girth of the graph $Z^*(R)$.

Let $M$ be an $R$-module. Consider $R(+)M = \{(a, m) : a \in R, m \in M\}$ and let $(a, m)$ and $(b, n)$ be two elements of $R(+)M$. Define: $(a, m) + (b, n) = (a + b, m + n)$ and $(a, m)(b, n) = (ab, am + bn)$. Under this definitions $R(+)M$ becomes a commutative ring with identity. Call this ring the idealization of $M$ in $R$ [5]. Let $M$ be an $R$-module. Then $M$ is called prime if whenever $rm = 0$ either $m = 0$ or $rM = 0$. So $M$ is prime if and only if for every non-zero submodule $N$ of $M$ we have $(0 :_R N) = (0 :_R M)$. In this case, $P = (0 :_R M)$ is a prime ideal of $R$ and we also say that $M$ is a $P$-prime $R$-module.

## 2 Diameter and Girth of $\Gamma(R(+)M)$

Let $M$ be a $P$-prime module over a Commutative ring $R$. The notation below will be kept in this paper: $V_1 = \{(0, m) : m \in M^*\}$, $V_2 = \{(a, n) : a \in P^*, n \in M\}$ and $V_3 = \{(a, n) : a \in Z^*(R), n \in M\}$.

**Proposition 2.1** Let $R$ be a commutative ring and let $M$ be a $P$-prime $R$-module. Then:

(i) If $P \neq 0$, then $(a, m) \in Z(R(+)M)$ if and only if $a \in P \cup Z(R)$.

(ii) If $P = 0$, then $(a, m) \in Z(R(+)M)$ if and only if $a = 0$ and $m \in M^*$.

**Proof.** (i) Let $(a, m) \in Z(R(+)M))$. We may assume that $a \neq 0$. There exist a non-zero element $(b, n) \in R(+)M$ such that $(a, m)(b, n) = (ab, an + bm) = (0, 0)$. If $b = 0$, then $a \in (0 :_R n) = P$; if $b \neq 0$, then $a \in Z(R)$. Conversely, assume that $(a, m) \in R(+)M$ with $a \in P \cup Z(R)$. If $a \in Z(R)$, then $ab = 0$ for some non-zero element $b \in R$. If $b \in P$, then $(a, m)(b, 0) = (0, 0)$. If $b \notin P$, then there is an element $x$ of $M$ such that $bx \neq 0$. Then $(a, m)(0, bx) = (0, 0)$. Finally, if $a \in P$, then there exists a non-zero element $y$ of $M$ such that $ay = 0$. Therefore, $(a, m)(0, y) = (0, 0)$, and so the proof is complete.
(ii) Let \((a, m) \in Z(R(+)M)\). We may assume that \(a \neq 0\). There exist a non-zero element \((b, n)\) of \(R(+)M\) such that \(ab = 0\) and \(an + bm = 0\). Since \(M\) is a 0-prime \(R\)-module, we must have \(R\) is an integral domain; hence if \(a \neq 0\), then \(b = 0\), \(n \neq 0\) and \(a \in (0 :_R n) = 0\) which is a contradiction. Therefore, \(a = 0\) and \(m \neq 0\) since \((a, m) \neq 0\). The other implication is clear.

**Theorem 2.2** Let \(R\) be a commutative ring and let \(M\) be a \(P\)-prime \(R\)-module. Then:

(i) If \(P = 0\), then \(Z(R(+)M)^* = V_1\).

(ii) If \(P \neq 0\) and \(Z(R)^* \neq \emptyset\), then \(Z(R(+)M)^* = V_1 \cup V_2 \cup V_3\).

(iii) If \(P \neq 0\) and \(Z(R)^* = \emptyset\), then \(Z(R(+)M)^* = V_1 \cup V_2\).

**Proof.** This follows from Proposition 2.1. □

**Theorem 2.3** Let \(M\) be a prime module over a commutative ring \(R\) and let \(\Gamma(R) \neq \emptyset\). Then \(\Gamma(R(+)M)\) is complete if and only if \(Z(R) \subseteq (0 :_R M)\).

**Proof.** Since \(\Gamma(R) \neq \emptyset\), we must have \((0 :_R M) = P \neq 0\). Assume \(\Gamma(R(+)M)\) is complete. Let \(r \in Z(R)\), \(0 \neq m \in M\). We may assume that \(r \neq 0\). Then Theorem 2.2 gives \((0, m), (r, 0) \in Z(R(+)M)^*\); hence \((r, 0)(0, m) = (0, 0)\). Therefore, \(r \in (0 :_R m) = P\). Conversely, assume that \(Z(R) \subseteq P\) and let \((a, m), (b, n) \in Z(R(+)M)^*\). If \(a = b = 0\), then clearly \((a, m)(b, n) = (0, 0)\).

If \(b = 0\) and \(a \in Z^*(R) \subseteq P\), then \(an = 0\); hence \((a, m)(b, n) = (0, 0)\).

If \(a, b \in Z^*(R) \subseteq P\), then \(an = 0 = bm\), so \((a, m)(b, n) = (0, 0)\). Thus \(\Gamma(R(+)M)\) is complete. □

Note that if \(M\) is a prime \(R\)-module, then any non-zero submodule of \(M\) is prime. Therefore, by Theorem 2.3, we have the following corollary:

**Corollary 2.4** Let \(R\) be a commutative ring, \(M\) a prime \(R\)-module, \(N\) a non-zero submodule of \(M\) and \(\Gamma(R) \neq \emptyset\). Then \(\Gamma(R(+)N)\) is complete if and only if \(\Gamma(R(+)N)\) is complete.

**Proposition 2.5** Let \(M\) be a \(P\)-prime module over a commutative ring \(R\) and let \(\Gamma(R) = \emptyset\). Then:

(i) If \(P = 0\), then \(\Gamma(R(+)M)\) is complete.

(ii) If \(P \neq 0\), then \(\text{diam}(\Gamma(R(+)M)) = 2\).

**Proof.** (i) Since \(\Gamma(R) = \emptyset\), we must have \(R\) is an integral domain. If \(P = 0\), then Theorem 2.2 gives \(Z(R(+)M)^* = V_1\), so clearly it is complete.

(ii) If \(P \neq 0\), then Theorem 2.2 gives \(Z(R(+)M)^* = V_1 \cup V_2\). Let \(z_1 = (a, m), z_2 = (b, n) \in Z(R(+)M)^*\). If \(z_1, z_2 \in V_1\), then \(z_1z_2 = 0\). If \(z_1 \in V_2\) and \(z_2 \in V_1\), then \(a \in P\) and \(b = 0\); hence \(z_1z_2 = 0\). Similarly, if \(z_1 \in V_1\) and \(z_2 \in V_2\), then \(z_1z_2 = 0\). So suppose that \(z_1, z_2 \in V_2\) and let \(0 \neq x \in M\). Then \(a, b \in P\); hence \(z_1 - (0, x) - z_2\) is a path, as required. □
Theorem 2.6 Let $M$ be a $P$-prime module over a commutative ring $R$ and let $\Gamma(R) = \emptyset$. Then $\text{diam}(\Gamma(R(+)M)) \leq 2$.

Proof. This follows from Proposition 2.5. □

Example 2.7 (i) Since $Z$ is a 0-prime $Z$-module, we must have $\Gamma(R(+)M)$ is complete by Theorem 2.5 (i).

(ii) Let $M = Z_3$ denote the ring of integers modulo 3. Then $M$ is a $3Z$-prime $Z$-module. Then $\text{diam}(\Gamma(R(+)M)) = 2$ by Theorem 2.5 (ii).

Let $R$ be a commutative ring and let $M$ be a $P$-prime $R$-module. If $|M| \geq 4$, then $g(\Gamma(R(+)M)) = 4$ by [2, p. 237]. Then we only need to consider when the $P$-prime module $M$ has two or three elements. Also, since the module $M$ is unitary, the ring $R$ cannot have fewer than three elements. So throughout this section we shall assume unless otherwise stated, that $|R| \geq 3$.

Lemma 2.8 Let $R$ be a commutative ring with identity and $M \cong Z_3$ a $P$-prime $R$-module. Then:

(i) $P \neq 0$ if and only if $|R| > 3$.

(ii) $P = 0$ if and only if $|R| = 3$.

Proof. (i) Since $P \neq 0$ and it is prime, we must have $|P| \geq 3$; hence $|R| \geq 4$. Conversely, assume that $|R| \geq 4$, so by [2, p. 237], there always exists a non-zero $r \in R$ such that $rZ_3 = 0$. Therefore, $P \neq 0$. (ii) Is clear. □

Theorem 2.9 Let $R$ be a commutative ring with identity and $M \cong Z_3$ a $P$-prime $R$-module. Then

(i) $g(\Gamma(R(+)M)) = 3$ if and only if $|R| > 3$

(ii) $g(\Gamma(R(+)M)) = \infty$ if and only if $|R| = 3$

Proof. This follows from Lemma 2.8 and [2, Theorem 2.1]. □

Corollary 2.10 Let $R$ be a commutative ring with identity and $M \cong Z_2$ a $P$-prime $R$-module. Then:

(i) If $P = 0$, then $g(\Gamma(R(+)M)) = \infty$.

(ii) If $P \neq 0$ and $\Gamma(R) = \emptyset$, then $g(\Gamma(R(+)M)) = \infty$.

(iii) If $Z^*(R) \subseteq P \neq 0$, $g(\Gamma(R)) = 3$ and $a^2 = 0$ for some $a \in P$, then $g(\Gamma(R(+)M)) = 3$.

Proof. This follows from [2, Theorem 2.2] and Proposition 2.1. □

Example 2.11 (i) Let $M = Z_3$ denote the ring of integers modulo 3. Then $M$ is a $3Z$-prime $Z$-module. Then $g(\Gamma(R(+)M)) = 3$ by Theorem 2.9 (i).

(ii) Let $M = Z_2$ denote the ring of integers modulo 2. Then $M$ is a $2Z$-prime $Z$-module. Then $g(\Gamma(R(+)M)) = \infty$ by Corollary 2.10 (ii).
References


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