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An Inversion Technique for ℓ_2-Transform with Applications

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Abstract

In this article,we derive a complex inversion formula and some new theorems related to ℓ_2-transform defined in [1],[2],[3].We also give an application for solution to non-homogeneous wave equation.

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1 Introduction

The Laplace-type integral transform called ℓ_2-transform where the ℓ_2-transform is defined as

$$\ell_2\{f(t);s\} = \int_0^\infty t\exp(-s^2t^2)f(t)dt. \tag{1-1}$$

If we make a change of variable in the right-hand side of the above integral (1-1), we get,

$$\ell_2\{f(t);s\} = \frac{1}{2}\int_0^\infty \exp(-ts^2)f(\sqrt{t})dt. \tag{1-2}$$

We have the following relationship between the Laplace-transform and the ℓ_2 -transform

$$\ell_2\{f(t);s\} = \frac{1}{2}L\{f(\sqrt{t});s^2\}. \tag{1-3}$$

First, we calculate ℓ_2-transform of some special functions.
Example 1.1 :

$$\ell_2\{\frac{\sin t^2}{t^2}\} = \int_0^\infty t\exp(-s^2t^2)\frac{\sin t^2}{t^2}dt = \frac{1}{2}\arctan\frac{1}{s^2}\ .$$

Setting $s = 0$ in the above integral, we obtain

$$\int_0^\infty \frac{\sin t^2}{t} dt = \frac{\pi}{4} \, .$$

Example 1.2 : Show that

$$\ell_2\{H(t-a); s\} = \frac{1}{2s^2} e^{-s^2 a^2} \, . \tag{1-4}$$

$$\ell_2\{H(t-a); s\} = \int_0^\infty t \exp(-s^2 t^2) H(t-a) dt = \int_a^\infty t \exp(-s^2 t^2) dt$$
$$= -\frac{1}{2s^2} \exp(-s^2 t^2)]_a^\infty = \frac{1}{2s^2} \exp(-s^2 a^2). \qquad (t > a)$$

Example 1.3 : Show that

$$\ell_2\{t^n; s\} = \frac{\Gamma(\frac{n}{2}+1)}{2s^{n+2}} \, . \tag{1-5}$$

Solution: By definition $\ell_2\{t^n; s\} = \int_0^\infty t^{n+1} \exp(-s^2 t^2) dt$,
the integral on the right-hand side may be evaluated by changing the variable of the integration from t to u where, $s^2 t^2 = u$

$$\ell_2\{t^n; s\} = \int_0^\infty (\frac{\sqrt{u}}{s})^{n+1} e^{-u} \frac{du}{2s\sqrt{u}} = \frac{1}{2s^{n+2}} \int_0^\infty u^{\frac{n}{2}} e^{-u} du, \tag{1-6}$$

using some Gamma function's relation in (1-6), we obtain

$$\ell_2\{t^n; s\} = \frac{\Gamma(\frac{n}{2}+1)}{2s^{n+2}} \, .$$

Example 1.4 : Show that

$$\ell_2\{e^{-at}; s\} = \frac{1}{2s^2} - \frac{a\sqrt{\pi}}{4s^3} \exp(\frac{a^2}{4s^2}) Erfc(\frac{a}{2s}) \, . \tag{1-7}$$

$$\ell_2\{e^{-at}; s\} = \int_0^\infty t \exp(-s^2 t^2 - at) dt = \int_0^\infty t \exp(-s^2(t^2 + \frac{at}{s^2})) dt$$
$$= \exp(\frac{a^2}{4s^2}) \int_0^\infty t \exp(-s^2(t + \frac{a}{2s^2})^2) dt,$$

in the right integral, let $s(t + \frac{a}{2s^2}) = u$ one has

$$\ell_2\{e^{-at}; s\} = \exp(\frac{a^2}{4s^2}) \int_{\frac{a}{2s}}^\infty (\frac{u}{s} - \frac{a}{2s^2}) \exp(-u^2) \frac{du}{s} =$$
$$\exp(\frac{a^2}{4s^2}) \{ \int_{\frac{a}{2s}}^\infty \frac{u}{s^2} \exp(-u^2) du - \frac{a}{2s^3} \int_{\frac{a}{2s}}^\infty \exp(-u^2) du \} =$$
$$\exp(\frac{a^2}{4s^2}) \frac{1}{2s^2} \exp(-\frac{a^2}{4s^2}) - \frac{a\sqrt{\pi}}{4s^3} Erfc(\frac{a}{2s}) =$$
$$\frac{1}{2s^2} - \frac{a\sqrt{\pi}}{4s^3} \exp(\frac{a^2}{4s^2}) Erfc(\frac{a}{2s}) \, .$$

2 Complex Inversion Formula for ℓ_2-transform

Main Theorem: let$F(\sqrt{s})$ is analytic function of s(assuming that $s = 0$ is not a branch point) except at finite number of poles each of which lies to the left of the vertical line Re $s = c$ and if $F(\sqrt{s}) \to 0$ as $s \to \infty$ through the left plane Re $s \leq c$, suppose that :

$$\ell_2\{f(t); s\} = \int_0^\infty t \exp(-s^2t^2)f(t)dt = F(s),$$

then

$$\ell_2^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{s^2t}dt = \sum_{k=1}^{m}[Res\{2F(\sqrt{s})e^{st^2}\}, s = s_k] \, .$$

Proof: First, it is not difficult to show that,

$$\ell_2^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{s^2t}dt,$$

but for second equality, we take a vertical closed semi-circle as contour of integration, and using residues theorem and boundedness of $F(\sqrt{s})$,it is possible to show that the result of the above theorem is valid.(in case of $s = 0$ a branch point, we need to take key-hole contour instead of simple vertical semi-circle). The above complex line integral is evaluated by using the inversion theorem. To see how it is done, we assume that $F(\sqrt{s})$ has a finite number of the poles in the left half plane Re $s \leq c$. We then let $\Gamma = \Gamma_1 + \Gamma_2$ be the closed contour consisting of the vertical line segment Γ_1 from $c - iR$ to $c + iR$ and vertical Semi circle Γ_2: $|s - c| = R$ lying to the left of vertical line Γ_1 . The radius R is to be taken large enough so that Γ encloses all the singularities of the function $F(\sqrt{s})$. Hence, by the residues theorem we have :

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR} 2F(\sqrt{s})e^{st^2}ds = \frac{1}{2\pi i}\int_{\Gamma} 2F(\sqrt{s})e^{st^2}ds - \frac{1}{2\pi i}\int_{\Gamma_2} 2F(\sqrt{s})e^{st^2}dt$$

$$= \sum_{k=1}^{n}[Res\{2F(\sqrt{s})e^{st^2}\}, s = s_k] - \frac{1}{2\pi i}\int_{\Gamma_2} 2F(\sqrt{s})e^{st^2}dt \, .$$

Where $s_1, s_2, ..., s_n$ are all the singularities of $F(\sqrt{s})$ in side the closed contour Γ . Taking the limit from both sides of the above relation as R tends to $+\infty$. It is not difficult to show by Jordan's lemma that,the second integral in the right tends to zero. We can use the above result, in the case of $F(\sqrt{s})$ has one branch point at $s = 0$, provided that, the contour is suitably modified. We consider the key - hole domain D consisting of the slit-plane $C - (-\infty, 0]$ satisfying $\varepsilon < |s| < R$,Re $s < c$, the residues theorem yields, $\oint_{\partial D} e^{st^2}F(\sqrt{s})ds = 0$, and the integral around ∂D breaks into the sums of six integrals, along the vertical line segment $[c-iR, c+iR]$,around the top circular contour Γ_R , along the top edge of the slit, around the small circle C_ε ,centered at origin and of radius ε , along the bottom edge of slit, and around the bottom edge of circular contour Γ_R.

Note: If the number of singularities is infinite, we take the semi-circles C_m which is centered at point c, with radius $R_m = \pi^2 m^2, m \in N$.

Example 2.1:By using complex inversion formula for ℓ_2-transform , show that

$$\ell_2^{-1}[\frac{1}{2s^2}\exp(-\frac{1}{4s^2})] = J_0(t)\ .$$

Solution: Let , $F(s) = \frac{1}{2s^2}\exp(-\frac{1}{4s^2})$ then, we have ,

$$2F(\sqrt{s}) = \frac{1}{2s}\exp(-\frac{1}{4s}),$$

therefore, $s = 0$ is a singular point (essential singularity not branch point). After using the above complex inversion formula, we obtain the original function as following,

$$f(t) = Res\{\frac{1}{s}\exp(-\frac{1}{4s})\exp(st^2), at\ s = 0\} = b_{-1}\ .$$

Where b_{-1} is the coefficient of the term $\frac{1}{s}$ in the Laurent expansion of $2F(\sqrt{s})\exp(st^2)$.Therefore we get the following relation,

$$2F(\sqrt{s})\exp(st^2) = \frac{1}{s}[1 + (st^2) + \frac{(st^2)^2}{2!} + \cdots][1 - \frac{1}{4s} + \frac{1}{(4s)^2 2!} - \frac{1}{(4s)^3 3!} + \cdots]\ .$$

From the above expansion we obtain,

$$f(t) = b_{-1} = [1 - \frac{t^2}{4^1 1!} + \frac{t^4}{4^2(2!)^2} - \frac{t^6}{4^3(3!)^2} + \cdots] = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots = J_0(t).$$

Theorem(Efros): Let $\ell_2\{f(t)\} = F(s)$ and assuming $\Phi(s), q(s)$ be analytic and such that , $\ell_2\{\phi(t,\tau)\} = \Phi(s)\tau e^{-\tau^2 q^2(s)}$, then one has,

$$\ell_2\{\int_0^\infty f(\tau)\phi(t,\tau)d\tau\} = F(q(s))\Phi(s)\ .$$

Proof: By definition of ℓ_2 -transform, one has

$$\ell_2\{\int_0^\infty f(\tau)\phi(t,\tau)d\tau\} = \int_0^\infty te^{-s^2t^2}(\int_0^\infty f(\tau)\phi(t,\tau)d\tau)dt,$$

changing the order of integration ,we get

$$\ell_2\{\int_0^\infty f(\tau)\phi(t,\tau)d\tau\} = \int_0^\infty f(\tau)dt(\int_0^\infty te^{-s^2t^2}\phi(t,\tau)d\tau)d\tau,$$

or,

$$\ell_2\{\int_0^\infty f(\tau)\phi(t,\tau)d\tau\} = \int_0^\infty f(\tau)\tau\Phi(s)e^{-\tau^2q^2(s)}d\tau\ .$$

Finally,

$$\ell_2\{\int_0^\infty f(\tau)\phi(t,\tau)d\tau\} = \Phi(s)\int_0^\infty f(\tau)\tau e^{-\tau^2q^2(s)}d\tau = \Phi(s)F(q(s))\ . \qquad Q.E.D$$

Example 2.2: Show that $\int_0^\infty\int_0^\infty te^{-t^2}J_0(x)\sin(xt)dxdt = \frac{\sqrt{\pi}}{2e}$.
Solution : Let, $\sin(xt) = \phi(x,t), J_0(x) = f(x)$ then we have the following ,

$$\ell_2\{\sin(xt), x\to s\} = \frac{\sqrt{\pi}}{4s^3}xe^{-(\frac{x}{2s})^2}, \qquad \ell_2\{J_0(x)\} = \frac{1}{2s^2}e^{-\frac{1}{4s^2}}\ .$$

Application of Efros theorem leads to the following ,

$$\ell_2\{\int_0^\infty J_0(t)\sin(xt)dt\} = \int_0^\infty\int_0^\infty te^{-s^2t^2}J_0(x)\sin(xt)dxdt = \Phi(s)F(q(s))$$

$$= \frac{\sqrt{\pi}}{4s^3}(\frac{4s^2}{2}e^{-s^2}),$$

or,

$$\int_0^\infty\int_0^\infty te^{-s^2t^2}J_0(x)\sin(xt)dxdt = \frac{\sqrt{\pi}}{2s}e^{-s^2}\ .$$

Setting $s = 1$,in the above relation, we get

$$\int_0^\infty\int_0^\infty te^{-t^2}J_0(x)\sin(xt)dxdt = \frac{\sqrt{\pi}}{2e}\ .$$

Some properties of the ℓ_2-transform

In this section we will recall some properties of the ℓ_2-transform that will be used to solve partial differential equation. First, we introduce a differential operator δ that we call the δ-derivative and define as

$$\delta_t = \frac{1}{t}\frac{d}{dt}\ . \tag{2-1}$$

We note that

$$\delta_t^2 = \delta_t\delta_t = \frac{1}{t^2}\frac{d^2}{dt^2} - \frac{1}{t^3}\frac{d}{dt}\ . \tag{2-2}$$

The δ -derivative operator can be successively applied in a similar fashion for any positive integer power.
Theorem2.1: If $f, f', \cdots, f^{(n-1)}$ are all continuous functions with a piecewise continuous derivative $f^{(n)}$ on the interval $t \geq 0$ and if all functions are of exponential order $\exp(c^2t^2)$ as $t\to\infty$ for some constant c then, [2]

1. For $n = 1, 2, \cdots$

$$\begin{aligned} \ell_2\{\delta_t^n f(t); s\} = 2^n s^{2n} \ell_2\{f(t); s\} - 2^{n-1} s^{2(n-1)} f(0^+) \\ -2^{n-2} s^{2(n-2)} (\delta_t f)(0^+) - \cdots - (\delta_t^{n-1} f)(0^+) \, . \end{aligned} \tag{2-3}$$

2. For $n = 1, 2, \cdots$

$$\ell_2\{t^{2n} f(t); s\} = \frac{(-1)^n}{2^n} \delta_s^n \ell_2\{f(t); s\} \, . \tag{2-4}$$

Lemma.2.1: If f is a continuous function with a piecewise continuous derivative f' on the interval $t \geq 0$ and if f, f' are of exponential $\exp(c^2 t^2)$as $t \to \infty$, where c is a constant, then

a)

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} 2s^2 F(s), \tag{2-5}$$

b)

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} 2s^2 F(s) \, . \tag{2-6}$$

Proof. a) We assume that, $\ell_2\{f(t); s\} = F(s)$,
using (2-3) we obtain

$$\ell_2\{\delta_t f(t); s\} = \int_0^\infty \exp(-s^2 t^2) f'(t) dt = 2s^2 F(s) - f(0), \tag{2-7}$$

if $f'(t)$ is piecewise continuous and of exponential order , we have

$$\lim_{s \to \infty} \int_0^\infty \exp(-s^2 t^2) f'(t) dt = 0 \, . \tag{2-8}$$

Then taking the limit as $s \to \infty$ in (2-8),

$$\lim_{s \to \infty} (2s^2 F(s) - f(0)) = 0, \tag{2-9}$$

or,

$$\lim_{s \to \infty} 2s^2 F(s) = f(0) = \lim_{t \to 0} f(t) \, . \tag{2-10}$$

Proof. b) Using (2-5), we have

$$\ell_2\{\delta_t f(t); s\} = \int_0^\infty \exp(-s^2 t^2) f'(t) dt = 2s^2 F(s) - f(0), \tag{2-11}$$

the limit of the left-hand side as $s \to 0$ is

$$\begin{aligned} \lim_{s \to 0} \int_0^\infty \exp(-s^2 t^2) f'(t) dt = \int_0^\infty f'(t) dt &= \lim_{p \to \infty} \int_0^p f'(t) dt \\ &= \lim_{p \to \infty} (f(p) - f(0)) \\ &= \lim_{t \to \infty} (f(t) - f(0)), \end{aligned} \tag{2-12}$$

the limit of the right-hand side as $s \to 0$

$$\lim_{s\to 0}(2s^2F(s) - f(0)) = \lim_{s\to 0} 2s^2F(s) - f(0)\,. \tag{2-13}$$

Thus,

$$\lim_{t\to\infty}(f(t) - f(0)) = \lim_{s\to 0} 2s^2F(s) - f(0),$$

or,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} 2s^2F(s)\,.$$

3 An alternative technique for solving partial differential equations

In this section, we introduce a new technique for solving non-homogeneous wave-equation by using ℓ_2-transform.

Example 3.1:

$$t\frac{\partial^2 u}{\partial t^2} - t^3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = xt^2\,. \qquad x, t > 0 \tag{3-1}$$

$$u(x,0) = 0, \qquad u(0,t) = 1, \qquad u_x(x,0) = 0$$

Solution : Dividing (3-1) by t^3 , we obtain

$$\frac{1}{t^2}\frac{\partial^2 u}{\partial t^2} - \frac{1}{t^3}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{x}{t}\,. \tag{3-2}$$

Using the definition of δ-derivative (1-7) and (1-8) we can express (3-2) as follows

$$\delta_t^2 u(x,t) = \frac{\partial^2 u}{\partial x^2} + \frac{x}{t}, \tag{3-3}$$

applying ℓ_2-transform to (3-3) we get

$$\ell_2\{\delta_t^2 u(x,t); s\} = \ell_2\{\frac{\partial^2 u}{\partial x^2} + \frac{x}{t}; s\}\,. \tag{3-4}$$

Using theorem 2.1, for $n = 2$ in (2-3) and performing some calculations we obtain the second- order differential equation

$$U_x''(x,s) - 4s^4U(x,s) = \frac{\sqrt{\pi}}{2s}x\,. \tag{3-5}$$

Where $U(x,s) = \ell_2\{u(x,t); s\}$.

Solving the second-order differential equation (3-5), we have

$$U(x,s) = c_1e^{2s^2x} + c_2e^{-2s^2x} - \frac{\sqrt{\pi}}{8s^5}x, \tag{3-6}$$

using lemma 2.1 and $u(0,t) = 1$, we obtain

$$c_1 = 0, \qquad c_2 = \frac{1}{2s^2}\,. \tag{3-7}$$

Substituting relation (3-6) into (3-5), we get

$$U(x,s) = \frac{1}{2s^2}e^{-2s^2x} - \frac{\sqrt{\pi}}{8s^5}x, \tag{3-8}$$

using the ℓ_2-inversion formula ,(3-8) yields, the formal solution,

$$u(x,t) = H(t - \sqrt{2x}) - \frac{xt^3}{3}\,.$$

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