Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 28, 1379 - 1386

Generalized Product Theorem for L_2-Transform with Applications

A. Aghili and A. Sedghi

Department of Mathematics
University of Guilan, P.O.Box 1841, Rasht, Iran
armanaghili@yahoo.com
sedghi1194@yahoo.com

Abstract

In this article , we prove some new theorems related to l_2-transform defined in [1],[2],[3] .We also give an application for solution to Fox singular integral equations with trigonometric kernel and the non-homogeneous systems of differential equations.

Mathematics Subject Classification: Primary 44A10 , Secondary 44A35

Keywords: l_2-transform, Laplace transform, Efros's theorem, Heavisides's function. Fox singular integral equations

1 Introduction

The Laplace-type integral transform called l_2-transform where the l_2-transform is defined as

$$l_2\{f(t);s\} = \int_0^\infty t\exp(-s^2t^2)f(t)\,dt \tag{1.1}$$

If we make a change of variable in the right-hand side of the above integral (1.1), we get,

$$l_2\{f(t);s\} = \frac{1}{2}\int_0^\infty e^{-ts^2} f(\sqrt{t})\,dt \tag{1.2}$$

we have the following relationship between the Laplace-transform and the l_2-transform

$$l_2\{f(t);s\} = \frac{1}{2}L\{f(\sqrt{t});s^2\} \tag{1.3}$$

First, we calculate l_2-transform of some special functions.

Example 1.1 – show that

1.

$$l_2\{H(t-a);s\} = \frac{1}{2s^2}e^{-s^2a^2} \tag{1.4}$$

2.

$$l_2\{t^n;s\} = \frac{\Gamma(\frac{n}{2}+1)}{2s^{n+2}} \tag{1.5}$$

Solution : see [1]

2 Complex Inversion Formula for l_2-transform

Theorem 1 (Main Theorem). *let* $F(\sqrt{s})$ *is analytic function of* s *(assuming that* $s=0$ *is not a branch point) except at finite number of poles each of which lies to the left of the vertical line* $\Re e\, s = c$ *and if* $F(\sqrt{s}) \to 0$ *as* $s \to \infty$ *through the left plane* $\Re e\, s \leqslant c$*, suppose that*

$$l_2\{f(t);s\} = \int_0^\infty t\exp(-s^2t^2)f(t)\,dt = F(s)$$

$$\text{Then } l_2^{-1}\{F(s)\} = f(t) = \tfrac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{st^2}\,dt$$

$$= \sum_{k=1}^{m}\left[\Re e\, s\{2F(\sqrt{s})e^{st^2}\}, s = s_k\right]$$

Proof. See [1] ☐

Example – By using complex inversion formula for l_2-Transform, show that

$$l_2^{-1}\left[\frac{1}{2s^2}\exp\frac{1}{4s^2}\right] = I_0(t)$$

where I_0 is modified Bessel function of first kind and order zero.

Solution: Let, $F(s) = \dfrac{1}{2s^2}\exp\left(\dfrac{1}{4s^2}\right)$ then, we have,

$$2F(\sqrt{s}) = \frac{1}{2s}\exp(\frac{1}{4s})$$

Therefore, $s = 0$ is a singular point (essential singularity not branch point). After using the above complex inversion formula, we obtain the original function as following,

$$f(t) = Res\left\{\frac{1}{s}\exp\left(\frac{1}{4s}\right)\exp st^2, \text{at } s = 0\right\} = b_{-1}$$

where, b_{-1} is the coefficient of the term $\frac{1}{s}$ in the Laurent expansion of $2F(\sqrt{s})\exp st^2$. Therefore we get the following relation,

$$2F(\sqrt{s})\exp st^2 = \frac{1}{s}\left[1 + (st^2) + \frac{(st^2)^2}{2!} + \cdots\right]\left[1 + \frac{1}{4s} + \frac{1}{(4s)^2 2!} + \frac{1}{(4s)^3 3!} + \cdots\right]$$

from the above expansion we obtain,

$$\begin{aligned} f(t) = b_{-1} = &\left[1 + \frac{t^2}{4^1(1!)} + \frac{t^4}{4^2(2!)} + \frac{t^6}{4^3(3!)} + \cdots\right] \\ &= 1 + \frac{t^2}{2^2} + \frac{t^4}{2^2 \times 4^2} + \frac{t^6}{2^2 \times 4^2 \times 6^2} + \cdots = I_0(t) \end{aligned}$$

Theorem 2.1 (Generalized product Theorem). *Let* $l_2(f(t)) = F(s)$ *and assuming* $\Phi(s)$, $q(s)$ *be analytic and such that,* $l_2(\Phi(t,\tau)) = \Phi(s)\tau e^{-\tau^2 q^2(s)}$, *then one has,*

$$l_2\left\{\int_0^\infty f(\tau)\Phi(t,\tau)d\tau\right\} = F(q(s))\Phi(s)$$

Proof. See [1] □

2.1 Solution to Fox - Singular Integral Equation with trigonometric kernel, using l_2-transform:

$$f(t) = g(t) + \lambda\int_0^\infty f(\tau)\varphi(t,\tau)d\tau \qquad (2.1)$$

Note: Let $l_2(f(t)) = F(s)$, $l_2(g(t)) = G(s)$ and assuming $\Phi(s)$, $q(s)$ be analytic and such that, $l_2(\varphi(t,\tau)) = \phi(s)\tau e^{-\tau^2 q^2(s)}$, and using the above theorem, then by taking l_2-transform of integral equation (2.1), leads to the following relation,

$$F(s) = G(s) + \lambda\Phi(s).F(q(s)) \qquad (2.2)$$

In case of trigonometric kernel, for example, $\varphi(t,\tau) = \sin(\tau t)$, we have

$$l_2[\sin(\tau t)] = \frac{t\sqrt{\pi}}{4s^3}e^{-\frac{t^2}{4s^2}} \rightarrow F(s) = G(s) + \lambda\Phi(s).F(\frac{1}{2s}) \qquad (2.3)$$

It is clear that, $q(s) = \frac{1}{2s}$, now in relation (2.2) we replace s by $\frac{1}{2s}$, we get

$$F(\frac{1}{2s}) = G(\frac{1}{2s}) + \lambda\Phi(\frac{1}{2s}).F(s) \tag{2.4}$$

combination of (2.3) and (2.4) and calculation of $F(s)$ leads to the following,

$$F(s) = \frac{G(s) + \lambda\Phi(s)G(\frac{1}{2s})}{1 - \lambda^2\Phi(s)\Phi(\frac{1}{2s})}, \tag{2.5}$$

relation (2.5) can be rewritten as (in terms of residue theorem for l_2-transform) follows,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2\frac{G(\sqrt{s}) + \lambda\Phi(\sqrt{s})G(\frac{1}{2\sqrt{s}})}{1 - \lambda^2\Phi(\sqrt{s})\Phi(\frac{1}{2\sqrt{s}})} e^{st^2}\, ds \tag{2.6}$$

Example 2.1 – Solve the following homogeneous singular integral equation.

$$\frac{2}{\pi}\int_0^\infty f(\tau)\sin(t\tau)\, d\tau = 1 \tag{2.7}$$

Solution: l_2-transform of the above integral equation, leads to the following

$$F(\frac{1}{2s})\frac{\sqrt{\pi}}{4s^3} = \frac{1}{2s^2} \tag{2.8}$$

or,

$$F(s) = \frac{\sqrt{\pi}}{2s} \quad \rightarrow \quad f(\tau) = l_2^{-1}[\frac{\sqrt{\pi}}{2s}] \quad \rightarrow \quad f(\tau) = \frac{1}{\tau}$$

2.2 Applications of generalized product theorem.

Example 2.2 – Show that $\int_0^\infty \sin x^2\, dx = \frac{\sqrt{2\pi}}{4}$ (Fresnel's Integral)

Solution: Let $I(t) = \int_0^\infty \sin tx^2\, dx$, then by introducing a change of variables $x^2 = u$ we obtain,

$$I(t) = \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{u}}\sin tu\, du \tag{2.9}$$

taking l_2-transform of the above relation and using generalized product theorem, we get

$$l_2\,[I(t)] = \frac{\Gamma(\frac{3}{4})}{2}\,(2s)^{\frac{3}{2}}(\frac{\sqrt{\pi}}{4s^3}) \tag{2.10}$$

after simplifying, one has

$$l_2\,[I(t)] = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\frac{\Gamma\left(\frac{3}{4}\right)}{2s^{\frac{3}{2}}} \quad , \tag{2.11}$$

upon inversion the above relation, we have

$$I(t) = l_2^{-1}\left[\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\frac{\Gamma(\frac{3}{4})}{2s^{\frac{3}{4}}}\right] = \sqrt{\frac{\pi}{2t}} \quad , \tag{2.12}$$

Proof. In (2.12) setting $t = 1$, we obtain $I(1) = \int_0^\infty \sin x^2\,dx = \frac{\sqrt{2\pi}}{4}$ □

Example 2.3 – Show that $\int_0^\infty \frac{\cos mx}{x}\sin xt\,dx = \frac{\pi}{2}H(t-m)$

Solution: let $I(t) = \int_0^\infty \frac{\cos mx}{x}\sin xt\,dx$, we calculate l_2-transform of the above relation by using main theorem and theorem 1, one has

$$l_2[I(t)] = (\sqrt{\pi s})e^{-m^2s^2}\frac{\sqrt{\pi}}{4s^3} \quad \Rightarrow \quad F(s) = \frac{\pi}{4s}e^{-m^2s^2} \quad \Rightarrow \quad 2F(\sqrt{s}) = \frac{\pi}{2s}e^{-m^2s}$$

finally, $I(t) = \Re\mathrm{e}\,s[2F(\sqrt{s})e^{st^2}\,, s=0] = \lim_{s\to 0}(\frac{\pi}{2}e^{-m^2s}e^{st^2}) = \frac{\pi}{2}H(t-m)$

2.3 Some elementary properties of the l_2-transform

In this section we will recall some elementary properties of the l_2-transform that will be used to solve systems of differential equations. First, we introduce a differential operator δ that we call the δ-derivative and define as

$$\delta_t = \frac{1}{t}\frac{d}{dt} \tag{2.13}$$

we note that

$$\delta_t^2 = \delta_t\delta_t = \frac{1}{t^2}\frac{d^2}{dt^2} - \frac{1}{t^3}\frac{d}{dt} \tag{2.14}$$

The δ-derivate operator can be successively applied in a similar fashion for any positive integer power.

Theorem 2.2. *If* $f, f', \cdots, f^{(n-1)}$ *are all continuous functions with a piecewise continuous derivate* $f^{(n)}$ *on the interval* $t \geqslant 0$*, and if all functions are of exponential order* $\exp(c^2t^2)$ *as* $t \to \infty$ *for some constant* c*, then,*

1.

$$\ell_2\left\{\delta_t^n f(t); s\right\} = 2^n s^{2n}\ell_2\left\{f(t); s\right\} - 2^{n-1}s^{2(n-1)}f(0^+) - 2^{n-2}s^{2(n-2)}(\delta_t f)(0^+) - \cdots - (\delta_t^{n-1}f)(0^+) \quad (2.15)$$

for $n = 1, 2, \cdots$

2.

$$\ell_2\left\{t^{2n}f(t); s\right\} = \frac{(-1)^n}{2^n}\delta_s^n \ell_2\left\{f(t); s\right\} \quad \textit{for} \quad n = 1, 2, 3, \cdots \quad (2.16)$$

Proof. See [2],[3] □

Lemma 2.1. *If* f *is continuous function with a piecewise continuous derivate* f' *on the interval* $t \geqslant 0$*, and if* f, f' *are of exponential* $\exp(c^2t^2)$ *as* $t \to \infty$*, where* c *is a constant, then*

1.

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} 2s^2F(s) \quad (2.17)$$

2.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} 2s^2F(s) \quad (2.18)$$

Proof. See [1] □

3 An alternative technique for solving systems of differential equations

In this section by some examples, we introduce a new technique for solving non homogeneous systems of differential equations using l_2-transform.

Example 3.1 : $\begin{cases} ty'' - y' = t^3x \\ tx'' - x' = t^3y \end{cases} \qquad x(0) = 1, \quad y(0) = 0$

$$\begin{cases}\frac{1}{t^2}y'' - \frac{1}{t^3}y' = x\\ \frac{1}{t^2}x'' - \frac{1}{t^3}x' = y\end{cases} \Rightarrow \begin{cases}\delta_t^2 y(t) = x(t)\\ \delta_t^2 x(t) = y(t)\end{cases} \Rightarrow \begin{cases}\ell_2\left\{\delta_t^2 y(t); s\right\} = \ell_2\left\{x(t); s\right\}\\ \ell_2\left\{\delta_t^2 x(t); s\right\} = \ell_2\left\{y(t); s\right\}\end{cases}$$

$$\begin{cases}4s^4Y(s) - 2s^2y(0) - (\delta_t y)(0) = X(s)\\ 4s^4X(s) - 2s^2x(0) - (\delta_t x)(0) = Y(s)\end{cases} \to \begin{cases}4s^4Y(s) = X(s)\\ 4s^4X(s) - 2s^2 = Y(s)\end{cases}$$

$$16s^8Y(s) - 2s^2 = Y(s) \Rightarrow Y(s) = \frac{2s^2}{16s^8-1} = \frac{s^2}{4s^4-1} - \frac{s^2}{4s^4+1}$$
$$y(t) = \frac{1}{2}\cosh\frac{1}{2}t^2 - \frac{1}{2}\cos\frac{1}{2}t^2$$
$$X(s) = \frac{8s^6}{16s^8-1} = \frac{s^2}{4s^4-1} + \frac{s^2}{4s^4+1} \to x(t) = \frac{1}{2}\cosh\frac{1}{2}t^2 + \frac{1}{2}\cos\frac{1}{2}t^2$$

Remark. We consider systems of non - homogeneous differential equations in general form as follows

$$\frac{1}{t}\overset{\rightharpoonup}{X}'(t) = A\overset{\rightharpoonup}{X}(t) + B \tag{3.1}$$

where, A and B are coefficient and constants matrices of type $(n \times n)$ and $(n \times 1)$ respectively and $\overset{\rightharpoonup}{X}'(t)$, $\overset{\rightharpoonup}{X}(t)$ are column vectors. In order to solve the above system,first, we take l_2-transform of (3.1) we get,

$$l_2\left[\frac{1}{t}\overset{\rightharpoonup}{X}'(t)\right] = Al_2\left[\overset{\rightharpoonup}{X}(t)\right] + Bl_2[1]$$

or,

$$2s^2l_2\left[\overset{\rightharpoonup}{X}'(t)\right] - \overset{\rightharpoonup}{X}(0) = Al_2\left[\overset{\rightharpoonup}{X}(t)\right] + \frac{1}{2s^2}B$$

if we simplify the above relation, we obtain

$$(2s^2I - A)l_2\left[\overset{\rightharpoonup}{X}(t)\right] = \overset{\rightharpoonup}{X}(0) + \frac{1}{2s^2}B$$

then,

$$l_2\left[\overset{\rightharpoonup}{X}(t)\right] = (2s^2I - A)^{-1}(\overset{\rightharpoonup}{X}(0) + \frac{1}{2s^2}B)$$

finally, $\overset{\rightharpoonup}{X}(t) = l_2^{-1}\left\{(2s^2I - A)^{-1}(\overset{\rightharpoonup}{X}(0) + \frac{1}{2s^2}B)\right\}$

References

[1] A. Aghili and A. Sedghi, An inversion technique for l_2-transform with applications, (2006).

[2] O. Yurekli and S. Wilson, A new method of solving Bessel's differential equation using the l_2-transform, *Appl. Math. Comput.,* **130** (2-3) (2002), 587 - 591.

[3] O. Yurekli and S. Wilson, A new method of solving Hermite's differential equation using the l_2-transform, *Appl. Math. Comput.,* **145** (2-3) (2003), 495 - 500.

Received: April 4, 2007