A Note on Fox-Singular Integral Equations and its Application

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Abstract

In this article, we derive some new theorems related to \(l_2\)-transform defined in [1],[2] we give also an application for solution to non-homogeneous Fox – singular integral equation. Finally, we prove also some important integral relations.

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1 Introduction

The Laplace-type integral transform called \(l_2\)-transform where the \(l_2\)-transform is defined as

\[
l_2\{f(t); s\} = \int_0^\infty t \exp(-s^2t^2) f(t) \, dt
\]  

(1.1)

If we make a change of variable in the right-hand side of the above integral (1.1), we get,

\[
l_2\{f(t); s\} = \frac{1}{2} \int_0^\infty e^{-ts^2} f(\sqrt{t}) \, dt
\]  

(1.2)

we have the following relationship between the Laplace-transform and the \(l_2\)-transform

\[
l_2\{f(t); s\} = \frac{1}{2} L\{f(\sqrt{t}); s^2\}
\]  

(1.3)

First, we calculate \(l_2\)-transform of some special functions.
Example 1.1 – show that

1. \[ l_2\{H(t - a); s\} = \frac{1}{2s^2}e^{-s^2a^2} \quad (1.4) \]

2. \[ l_2\{t^n; s\} = \frac{\Gamma(\frac{n}{2} + 1)}{2s^{n+2}} \quad (1.5) \]

3. \[ l_2\{\text{erf}(at); s\} = \frac{a}{2s^2\sqrt{s^2 + a^2}} \quad (1.6) \]

4. \[ l_2\{\delta(t - a); s\} = ae^{-s^2a^2} \quad (1.7) \]

Solution: see [2]

Lemma 1.1. Show that,

\[ l_2(\ln t) = -\frac{\gamma + \ln s^2}{2s^2} \quad \gamma = \text{Euler constant} \]

Proof: By definition, we have

\[ l_2\{t^n; s\} = \frac{\Gamma(\frac{n}{2} + 1)}{2s^{n+2}} \]

or,

\[ l_2\{t^n; s\} = \int_0^\infty t \exp(-s^2t^2)t^\lambda \, dt = \frac{\Gamma(\frac{\lambda}{2} + 1)}{2s^{\lambda+2}} \]

If we differentiate the above relation w.r.r. \( \lambda \) (using Leibnitz’s rule), we get

\[ \int_0^\infty t \exp(-s^2t^2)t^\lambda \ln t \, dt = \frac{d}{d\lambda} \left[ \frac{\Gamma(\frac{\lambda}{2} + 1)}{2s^{\lambda+2}} \right] \]

or,

\[ \int_0^\infty t \exp(-s^2t^2)t^\lambda \ln t \, dt = \frac{1}{2s^2} \left[ \frac{\Gamma\left(\frac{\lambda+2}{2}\right)}{s^{2\lambda}} \right] \left[ \frac{\Gamma\left(\frac{\lambda+2}{2}\right)}{\Gamma\left(\frac{\lambda+2}{2}\right)} - 2 \ln s \right] \quad (1.8) \]
at this point, if we set $\lambda = 0$ and assuming $\Gamma'(1) = -\gamma$, we get
\[
\int_0^\infty t \exp(-s^2 t^2) \ln t \, dt = l_2[\ln t] = -\frac{\gamma + \ln s^2}{2s^2}
\]

Note: In relation (1.7) if we set $\lambda = -1, s = 1$, we obtain the following integral
\[
\int_0^\infty e^{-t^2} \ln t \, dt = \frac{1}{2} \Gamma' \left( \frac{1}{2} \right)
\]
In order to calculate $\Gamma' \left( \frac{1}{2} \right)$, we use the following well known identity
\[
\frac{\Gamma'(z + 1)}{\Gamma(z + 1)} = -\gamma + \sum_{n=1}^{+\infty} \left[ \frac{1}{n} - \frac{1}{n + z} \right]
\]
Setting $z = -\frac{1}{2}$, we obtain
\[
\frac{\Gamma' \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} = -\gamma + \left( \sum_{n=1}^{+\infty} \frac{2}{2n} - 2 \right)
\]
or,
\[
\frac{\Gamma' \left( \frac{1}{2} \right)}{\sqrt{\pi}} = -(\gamma + 2 \ln 2)
\]
finally,
\[
\int_0^{+\infty} e^{-t^2} \ln t \, dt = \frac{1}{2} \Gamma' \left( \frac{1}{2} \right) = -\frac{\sqrt{\pi}}{2} (\gamma + 2 \ln 2)
\]

**Lemma 1.2.** If $l_2[f(t)] = F(s)$, then $l_2[f^{2n} f(t)] = \frac{(-1)^n F^{(n)}(s)}{(2s)^{2n}}$

**Proof.** By definition,
\[
F(s) = l_2 \{ f(t); s \} = \int_0^\infty t \exp(-s^2 t^2) f(t) \, dt
\]
successive $n$-times differentiation w.r.t parameter $s$, and simplifying, leads to the following
\[
l_2 \left[ f^{2n} f(t) \right] = \frac{(-1)^n F^{(n)}(s)}{(2s)^{2n}}
\]
2 Complex Inversion Formula for $l_2$-transform

Theorem 1 (Main Theorem). let $F(\sqrt{s})$ is analytic function of $s$ (assuming that $s = 0$ is not a branch point) except at finite number of poles each of which lies to the left of the vertical line $\Re \; s = c$ and if $F(\sqrt{s}) \to 0$ as $s \to \infty$ through the left plane $\Re \; s \leq c$, suppose that

$$l_2\{f(t); s\} = \int_0^\infty t \exp(-s^2 t^2) f(t) \, dt = F(s)$$

Then $l_2^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{s t^2} \, dt$

$$= \sum_{k=1}^m [\Re \; s\{2F(\sqrt{s})e^{s t^2}\}, \; s = s_k]$$

Proof. See [1]

Theorem 2.1 (Generalized product Theorem – Efros Theorem [1]).

Let $l_2(f(t)) = F(s)$ and assuming $\Phi(s), q(s)$ be analytic and such that, $l_2(\Phi(t, \tau)) = \Phi(s)\tau e^{-\tau^2 q(s)}$, then one has,

$$l_2\left\{\int_0^\infty f(\tau)\Phi(t, \tau) \, d\tau\right\} = F(q(s))\Phi(s)$$

Proof. See [1]

Example 2.1 – Solve the following singular Integral equation.

$$f(t) = g(t) + \lambda \int_0^\infty f(\tau)\varphi(t, \tau) \, d\tau \quad (2.1)$$

Let $l_2(f(t)) = F(s)$, $l_2(g(t)) = G(s)$ and assuming $\Phi(s)\varphi(s)$ be analytic and such that $l_2(\phi(t, \tau)) = \Phi(s)\tau e^{-\tau^2 \varphi(s)}$, using the above theorem, then, by taking $l_2$-transform of integral equation (2.1), we obtain

$$F(s) = G(s) + \lambda \Phi(s).F(q(s)) \quad (2.2)$$

In case of trigonometric kernel, for example, $\varphi(t, \tau) = \sin(t\tau)$, we have

$$l_2[\sin(t\tau)] = \frac{t\sqrt{\pi}}{4s^3} e^{-\frac{t^2}{4s^2}} \to F(s) = G(s) + \lambda \Phi(s).F\left(\frac{1}{2s}\right) \quad (2.3)$$

It is clear that, $q(s) = \frac{1}{2s}$, now, in relation (2.2) we replace $s$ by $\frac{1}{2s}$, to get

$$F\left(\frac{1}{2s}\right) = G\left(\frac{1}{2s}\right) + \lambda \Phi\left(\frac{1}{2s}\right).F\left(\frac{s}{2}\right) \quad (2.4)$$
combination of (2.3) and (2.4) and calculation of \( F(s) \) leads to the following,

\[
F(s) = \frac{G(s) + \lambda \Phi(s) G \left( \frac{1}{2s} \right)}{1 - \lambda^2 \Phi(s) \Phi \left( \frac{1}{2s} \right)}
\]  

(2.5)

Relation (2.5) can be rewritten as (in term of residue theorem for \( l_2 \)-transform) follows,

\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2 \left( \frac{G(\sqrt{s}) + \lambda \Phi(\sqrt{s}) G \left( \frac{1}{2\sqrt{s}} \right)}{1 - \lambda^2 \Phi(\sqrt{s}) \Phi \left( \frac{1}{2\sqrt{s}} \right)} e^{st^2} \right) ds
\]  

(2.6)

**Example 2.2** – Solve the following non-homogeneous Fox-Singular Integral equation.

\[
f(x) = \frac{1}{x} + \lambda \int_0^\infty f(t) \sin xt \, dt \quad (x > 0, \, \lambda \neq \sqrt{\frac{2}{\pi}})
\]

Solution: upon using (2.5), one has

\[
F(s) = \frac{\sqrt{\pi} \sqrt{s} + \lambda \sqrt{\pi} \sqrt{s} \pi s}{1 - \lambda^2 \pi \sqrt{s} \pi s}
\]

or,

\[
F(s) = \frac{\sqrt{\pi} \sqrt{s} + \lambda \sqrt{\pi} \sqrt{s} \pi s}{1 - \lambda^2 \pi} = \frac{2}{2 - \lambda^2 \pi} \left\{ \frac{\sqrt{\pi} \pi \sqrt{s} + \lambda \pi}{4s^3} \right\}
\]

taking the inverse \( l_2 \)-transform, to get

\[
f(x) = \frac{2}{2 - \lambda^2 \pi} \left\{ \frac{1}{x} + \lambda \pi \right\}
\]

**Example 2.3** – Solve the integral equation, \( \frac{2}{\pi} \int_0^\infty \phi(x) \sin xt \, dx = \text{erf} \left( \frac{t}{2a} \right) \)

Solution: On using \( l_2 \)-transform followed by generalized product theorem and example 1.1 (part -3), yields

\[
\frac{2}{\pi} \Phi \left( \frac{1}{2s} \right) \frac{\sqrt{\pi}}{4s^3} = \frac{1}{2s^2 \sqrt{1 + a^2 s^2}}
\]

or,

\[
\Phi \left( \frac{1}{2s} \right) = \frac{\sqrt{\pi}}{4} \frac{1}{\sqrt{s^2 + a^2}}
\]
replacing \( s \) by \( \frac{1}{2s} \) in the above relationship, we get

\[
\Phi(s) = \frac{1}{2} \sqrt{\frac{\pi}{s^2 + a^2}}
\]

It is not difficult to show that,

\[
l_2 \left[ \frac{e^{-at^2}}{t} \right] = \frac{1}{2} \sqrt{\frac{\pi}{s^2 + a^2}}
\]

therefore,

\[
\phi(x) = \frac{e^{-a^2x^2}}{x}
\]

**Example 2.4** – Let us apply the generalized product theorem to show that

\[
I(m, \lambda) = \int_0^{+\infty} \frac{\cos mx}{x} \sin \lambda x \, dx = \frac{\pi}{2} H(\lambda - m)
\]

Solution: Let us assume that, \( I(m, \lambda) = \int_0^{+\infty} \frac{\cos mx}{x} \sin \lambda x \, dx \) on using \( l_2 \)-transform followed by generalized product theorem and theorem (2.2) yields,

\[
l_2\{I(m, \lambda)\} = l_2 \left[ \int_0^{+\infty} \frac{\cos mx}{x} \sin \lambda x \, dx \right] = \left( \frac{\sqrt{\pi}}{2s} e^{-\frac{m^2}{4s^2}} \right) \frac{\sqrt{\pi}}{4s^3} = \frac{\pi}{4s^2} e^{-m^2 s^2}
\]

On taking inverse \( l_2 \)-transform of the above relationship and using example 1.1 (part 1), we obtain

\[
I(m, \lambda) = \int_0^{+\infty} \frac{\cos mx}{x} \sin \lambda x \, dx = \frac{\pi}{2} H(\lambda - m)
\]

**Lemma 2.1.** show that \( \int_0^{\infty} x^{\mu-1} \sin tx \, dx = t^\mu \Gamma(\mu) \sin \left( \frac{\pi \mu}{2} \right) \)

**Proof.** Let us assume for the moment that,

\[
I(t) = \int_0^{\infty} x^{\mu-1} \sin tx \, dx
\]

\( l_2 \)-transform of the above relation followed by generalized product theorem, leads to

\[
l_2[I(t)] = l_2 \left[ \int_0^{\infty} x^{\mu-1} \sin tx \, dx \right] = \left( \frac{\Gamma \left( \frac{\mu+1}{2} \right)}{2s^{\mu+1}} \right) \frac{\sqrt{\pi}}{4s^3} = \frac{\Gamma \left( \frac{\mu+1}{2} \right) \sqrt{\pi}}{2^{2-\mu} s^{2-\mu}}
\]
or,

\[ l_2[I(t)] = \frac{\Gamma \left( \frac{\mu+1}{2} \right) \sqrt{\pi}}{2^{\mu} s^{2-\mu}} \]

now, we multiply the above relationship by, \( \frac{\Gamma(\frac{\mu}{2}) \Gamma(1-\frac{\mu}{2})}{\Gamma(\frac{\mu+1}{2}) \Gamma(1-\frac{\mu+1}{2})} \) to get

\[ l_2[I(t)] = \frac{\Gamma \left( \frac{\mu+1}{2} \right) \sqrt{\pi} \Gamma \left( \frac{\mu}{2} \right) \Gamma \left( 1-\frac{\mu}{2} \right)}{2^{1-\mu} s^{2-\mu} \Gamma \left( \frac{\mu}{2} \right) \Gamma \left( 1-\frac{\mu}{2} \right)} \]

at this point, we recall that two useful well-known identities as following,

1. \( \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \alpha \pi}, \quad (\alpha \neq \mathbb{Z}) \)

2. \( \Gamma(x)\Gamma(x + \frac{1}{2}) = 2^{1-2x}\sqrt{\pi} \Gamma(2x) \) (Legendre’s Duplication Formula)

using first relation leads to,

\[ l_2[I(t)] = \frac{\sqrt{\pi} \sin \frac{\mu \pi}{2}}{\pi} \cdot \frac{\Gamma(\frac{\mu}{2}) \Gamma(\frac{1+\mu}{2})}{2^{1-\mu} \pi} l_2[t^{-\mu}] \]

Now, if we use Legendre’s Duplication Formula and simplifying the above relation we get,

\[ I(t) = \frac{\sqrt{\pi} \sin \frac{\mu \pi}{2}}{\pi} \cdot \frac{\Gamma(\mu)2^{1-\mu} \sqrt{\pi}}{2^{1-\mu}} t^{-\mu} \]

Thus,

\[ I(t) = \sin \frac{\mu \pi}{2} \Gamma(\mu) t^{-\mu} \]

Note: In case of \( \mu = \frac{1}{2}, \mu = -\frac{1}{2} \) we obtain the following integrals,

\[ \mu = \frac{1}{2} \Rightarrow \int_{0}^{+\infty} \frac{\sin tx}{\sqrt{x}} \, dx = \sqrt{\frac{\pi}{2t}} \]

\[ \mu = -\frac{1}{2} \Rightarrow \int_{0}^{+\infty} \frac{\sin tx}{x \sqrt{x}} \, dx = \sqrt{2t \pi} \]

**Corollary 2.1.** we have,

\[ \int_{0}^{+\infty} \sin \lambda x^p \, dx = \frac{\Gamma(\lambda p)}{p \lambda^p} \sin \frac{\pi}{2p} \quad (p > 1) \]
Proof. Let us assume that

\[ I(\lambda) = \int_{0}^{+\infty} \sin \lambda x^p \, dx \]

If we set \( x^p = u \) in the above relation, after simplifying, one has

\[ I(\lambda) = \frac{1}{p} \int_{0}^{+\infty} u^{\frac{1}{p} - 1} \sin \lambda u \, du \]

If we use lemma 2.1, we get

\[ I(\lambda) = \frac{1}{p} \int_{0}^{+\infty} u^{\frac{1}{p} - 1} \sin \lambda u \, du = \frac{1}{p} \lambda^{-\frac{1}{p}} \Gamma \left( \frac{1}{p} \right) \sin \frac{\pi}{2p} \]

or,

\[ I(\lambda) = \frac{\Gamma \left( \frac{1}{p} \right) \sin \frac{\pi}{2p}}{p \lambda^{\frac{1}{p}}} \]

\[ \square \]

References


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