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A Note on Fox-Singular Integral Equations and its Application

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Abstract

In this article, we derive some new theorems related to l_2-transform defined in [1],[2] we give also an application for solution to non-homogeneous Fox – singular integral equation. Finally, we prove also some important integral relations.

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1 Introduction

The Laplace-type integral transform called l_2-transform where the l_2-transform is defined as

$$l_2\{f(t);s\} = \int_0^\infty t\exp(-s^2t^2)f(t)\,dt \tag{1.1}$$

If we make a change of variable in the right-hand side of the above integral (1.1), we get,

$$l_2\{f(t);s\} = \frac{1}{2}\int_0^\infty e^{-ts^2} f(\sqrt{t})\,dt \tag{1.2}$$

we have the following relationship between the Laplace-transform and the l_2-transform

$$l_2\{f(t);s\} = \frac{1}{2}L\{f(\sqrt{t});s^2\} \tag{1.3}$$

First, we calculate l_2-transform of some special functions.

Example 1.1 – show that

1.

$$l_2\{H(t-a);s\} = \frac{1}{2s^2}e^{-s^2a^2} \tag{1.4}$$

2.

$$l_2\{t^n;s\} = \frac{\Gamma(\frac{n}{2}+1)}{2s^{n+2}} \tag{1.5}$$

3.

$$l_2\{\mathrm{erf}(at);s\} = \frac{a}{2s^2\sqrt{s^2+a^2}} \tag{1.6}$$

4.

$$l_2\{\delta(t-a);s\} = ae^{-s^2a^2} \tag{1.7}$$

Solution : see [2]

Lemma 1.1. *Show that,*

$$l_2(\ln t) = -\frac{\gamma + \ln s^2}{2s^2} \quad \gamma = \textit{Euler constant} \ .$$

Proof. : By definition, we have

$$l_2\{t^n;s\} = \frac{\Gamma(\frac{n}{2}+1)}{2s^{n+2}}$$

or,

$$l_2\{t^n;s\} = \int_0^\infty t\exp(-s^2t^2)t^\lambda\,dt = \frac{\Gamma(\frac{\lambda}{2}+1)}{2s^{\lambda+2}}$$

If we differentiate the above relation w.r.r. λ (using Leibnitz's rule), we get

$$\int_0^\infty t\exp(-s^2t^2)t^\lambda \ln t\,dt = \frac{d}{d\lambda}\left[\frac{\Gamma(\frac{\lambda}{2}+1)}{2s^{\lambda+2}}\right]$$

or,

$$\int_0^\infty t\exp(-s^2t^2)t^\lambda \ln t\,dt = \frac{1}{2s^2}\left[\frac{\Gamma(\frac{\lambda+2}{2})}{s^{2\lambda}}\right]\left[\frac{\Gamma'(\frac{\lambda+2}{2})}{\Gamma(\frac{\lambda+2}{2})} - 2\ln s\right] \tag{1.8}$$

at this point, if we set $\lambda = 0$ and assuming $\Gamma'(1) = -\gamma$, we get

$$\int_0^\infty t\exp(-s^2t^2)\ln t\,dt = l_2[\ln t] = -\frac{\gamma + \ln s^2}{2s^2}$$

Note: In relation (1.7) if we set $\lambda = -1$, $s = 1$, we obtain the following integral

$$\int_0^\infty e^{-t^2}\ln t\,dt = \frac{1}{2}\Gamma'\left(\frac{1}{2}\right)$$

In order to calculate $\Gamma'\left(\frac{1}{2}\right)$, we use the following well known identity

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = -\gamma + \sum_{n=1}^{+\infty}\left[\frac{1}{n} - \frac{1}{n+z}\right]$$

Setting $z = -\frac{1}{2}$, we obtain

$$\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = -\gamma + \left(\sum_{n=1}^{+\infty}\frac{2}{2n} - \frac{2}{2n-1}\right)$$

or ,

$$\frac{\Gamma'\left(\frac{1}{2}\right)}{\sqrt{\pi}} = -(\gamma + 2\ln 2)$$

finally,

$$\int_0^{+\infty} e^{-t^2}\ln t\,dt = \frac{1}{2}\Gamma'\left(\frac{1}{2}\right) = -\frac{\sqrt{\pi}}{2}(\gamma + 2\ln 2)$$

□

Lemma 1.2. *If* $l_2[f(t)] = F(s)$, *then* $l_2[f^{2n}f(t)] = \frac{(-1)^nF^{(n)}(s)}{(2s)^{2n}}$

Proof. By definition,

$$F(s) = l_2\left\{f(t);s\right\} = \int_0^\infty t\exp(-s^2t^2)f(t)\,dt$$

successive n-times differentiation w.r.t parameter s, and simplifying, leads to the following

$$l_2\left[t^{2n}f(t)\right] = \frac{(-1)^nF^{(n)}(s)}{(2s)^{2n}}$$

□

2 Complex Inversion Formula for l_2-transform

Theorem 1 (Main Theorem). *let* $F(\sqrt{s})$ *is analytic function of* s *(assuming that* $s = 0$ *is not a branch point) except at finite number of poles each of which lies to the left of the vertical line* $\Re e\, s = c$ *and if* $F(\sqrt{s}) \to 0$ *as* $s \to \infty$ *through the left plane* $\Re e\, s \leqslant c$*, suppose that*

$$l_2\{f(t); s\} = \int_0^\infty t \exp(-s^2t^2) f(t)\, dt = F(s)$$

$$\text{Then } l_2^{-1}\{F(s)\} = f(t) = \tfrac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s})e^{st^2}\, dt$$

$$= \sum_{k=1}^{m}\left[\Re e\, s\{2F(\sqrt{s})e^{st^2}\}, s = s_k\right]$$

Proof. See [1] □

Theorem 2.1 (Generalized product Theorem – Efros Theorem [1]). *Let* $l_2(f(t)) = F(s)$ *and assuming* $\Phi(s)$, $q(s)$ *be analytic and such that,* $l_2(\Phi(t,\tau)) = \Phi(s)\tau e^{-\tau^2 q^2(s)}$*, then one has,*

$$l_2\left\{\int_0^\infty f(\tau)\Phi(t,\tau)d\,\tau\right\} = F(q(s))\Phi(s)$$

Proof. See [1] □

Example 2.1 – Solve the following singular Integral equation.

$$f(t) = g(t) + \lambda\int_0^\infty f(\tau)\varphi(t,\tau)\, d\tau \tag{2.1}$$

Let $l_2(f(t) = F(s)$, $l_2(g(t)) = G(s)$ and assuming $\Phi(s)$,$q(s)$ be analytic and such that $l_2(\phi(t,\tau)) = \Phi(s)\tau e^{-\tau^2 q^2(s)}$,using the above theorem, then, by taking l_2-transform of integral equation (2.1), we obtain

$$F(s) = G(s) + \lambda\Phi(s).F(q(s)) \tag{2.2}$$

In case of trigonometric kernel, for example, $\varphi(t,\tau) = \sin(t\tau)$,we have

$$l_2[\sin(t\tau)] = \frac{t\sqrt{\pi}}{4s^3}e^{-\frac{t^2}{4s^2}} \to F(s) = G(s) + \lambda\Phi(s).F\left(\frac{1}{2s}\right) \tag{2.3}$$

It is clear that, $q(s) = \frac{1}{2s}$, now, in relation (2.2) we replace s by $\frac{1}{2s}$, to get

$$F\left(\frac{1}{2s}\right) = G\left(\frac{1}{2s}\right) + \lambda\Phi\left(\frac{1}{2s}\right).F\left(s\right) \tag{2.4}$$

combination of (2.3) and (2.4) and calculation of $F(s)$ leads to the following,

$$F(s)=\frac{G(s)+\lambda\Phi(s)G\left(\frac{1}{2s}\right)}{1-\lambda^2\Phi(s)\Phi\left(\frac{1}{2s}\right)} \tag{2.5}$$

Relation (2.5) can be rewritten as (in term of residue theorem for l_2-transform) follows,

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2\left(\frac{G(\sqrt{s})+\lambda\Phi(\sqrt{s})G\left(\frac{1}{2\sqrt{s}}\right)}{1-\lambda^2\Phi(\sqrt{s})\Phi\left(\frac{1}{2\sqrt{s}}\right)}e^{st^2}\right)ds \tag{2.6}$$

Example 2.2 – Solve the following non-homogeneous Fox-Singular Integral equation.

$$f(x)=\frac{1}{x}+\lambda\int_0^\infty f(t)\sin xt\,dt \qquad (x>0,\ \lambda\neq\sqrt{\frac{2}{\pi}})$$

Solution: upon using (2.5), one has

$$F(s)=\frac{\frac{\sqrt{\pi}}{2s}+\lambda\frac{\sqrt{\pi}}{4s^3}\sqrt{\pi}s}{1-\lambda^2\frac{\sqrt{\pi}}{4s^3}\frac{\sqrt{\pi}}{4(2s)^{-3}}}$$

or,

$$F(s)=\frac{\frac{\sqrt{\pi}}{2s}+\lambda\frac{\sqrt{\pi}}{4s^2}}{1-\frac{\lambda^2\pi}{2}}=\frac{2}{2-\lambda^2\pi}\left\{\frac{\sqrt{\pi}}{2s}+\lambda\frac{\pi}{4s^2}\right\}$$

taking the inverse l_2-transform, to get

$$f(x)=\frac{2}{2-\lambda^2\pi}\left\{\frac{1}{x}+\lambda\frac{\pi}{2}\right\}$$

Example 2.3 – Solve the integral equation, $\frac{2}{\pi}\int_0^\infty \phi(x)\sin xt\,dx=\operatorname{erf}(\frac{t}{2a})$

Solution: On using l_2-transform followed by generalized product theorem and example 1.1 (part -3), yields

$$\frac{2}{\pi}\Phi\left(\frac{1}{2s}\right)\frac{\sqrt{\pi}}{4s^3}=\frac{1}{2s^2\sqrt{1+a^2s^2}}$$

or,

$$\Phi\left(\frac{1}{2s}\right)=\frac{\sqrt{\pi}}{4}\frac{1}{\sqrt{s^2+a^2}}$$

replacing s by $\frac{1}{2s}$ in the above relationship, we get

$$\Phi(s) = \frac{1}{2}\sqrt{\frac{\pi}{s^2 + a^2}}$$

It is not difficult to show that,

$$l_2\left[\frac{e^{-a^2t^2}}{t}\right] = \frac{1}{2}\sqrt{\frac{\pi}{s^2 + a^2}}$$

therefore,

$$\phi(x) = \frac{e^{-a^2x^2}}{x}$$

Example 2.4 – Let us apply the generalized product theorem to show that

$$I(m,\lambda) = \int_0^{+\infty} \frac{\cos mx}{x} \sin \lambda x \, dx = \frac{\pi}{2} H(\lambda - m)$$

Solution: Let us assume that, $I(m,\lambda) = \int_0^{+\infty} \frac{\cos mx}{x} \sin \lambda x \, dx$ on using l_2-transform followed by generalized product theorem and theorem (2.2) yields,

$$l_2\{I(m,\lambda)\} = l_2\left[\int_0^{+\infty} \frac{\cos mx}{x} \sin \lambda x \, dx\right] = \left(\frac{\sqrt{\pi}}{2s} e^{-\frac{m^2}{4s^2}}\right)_{s\to\frac{1}{2s}} \frac{\sqrt{\pi}}{4s^3} = \frac{\pi}{4s^2} e^{-m^2s^2}$$

On taking inverse l_2-transform of the above relationship and using example 1.1 (part 1), we obtain

$$I(m,\lambda) = \int_0^{+\infty} \frac{\cos mx}{x} \sin \lambda x \, dx = \frac{\pi}{2} H(\lambda - m)$$

Lemma 2.1. *show that* $\int_0^{\infty} x^{\mu-1} \sin tx \, dx = t^{\mu}\Gamma(\mu) \sin\left(\frac{\pi\mu}{2}\right)$

Proof. Let us assume for the moment that,

$$I(t) = \int_0^{\infty} x^{\mu-1} \sin tx \, dx$$

l_2-transform of the above relation followed by generalized product theorem, leads to

$$l_2[I(t)] = l_2\left[\int_0^{\infty} x^{\mu-1} \sin tx \, dx\right] = \left(\frac{\Gamma\left(\frac{\mu+1}{2}\right)}{2s^{\mu+1}}\right)_{s\to\frac{1}{2s}} \left(\frac{\sqrt{\pi}}{4s^3}\right) = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\sqrt{\pi}}{2^{2-\mu}s^{2-\mu}}$$

or,

$$l_2[I(t)] = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\sqrt{\pi}}{2^{2-\mu}s^{2-\mu}}$$

now, we multiply the above relationship by, $\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(1-\frac{\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(1-\frac{\mu}{2}\right)}$ to get

$$l_2[I(t)] = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\sqrt{\pi}}{2^{1-\mu}s^{2-\mu}}\,\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(1-\frac{\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(1-\frac{\mu}{2}\right)}$$

at this point, we recall that two useful well-known identities as following,

1. $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin\alpha\pi}, \quad (\alpha \neq \mathbb{Z})$
2. $\Gamma(x)\Gamma(x+\frac{1}{2}) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x)$ (Legendre's Duplication Formula)

using first relation leads to,

$$l_2[I(t)] = \frac{\sqrt{\pi}\sin\frac{\mu\pi}{2}}{\pi}.\frac{\Gamma(\frac{\mu}{2})\Gamma(\frac{1+\mu}{2})}{2^{1-\mu}}l_2[t^{-\mu}]$$

Now, if we use Legendre's Duplication Formula and simplifying the above relation we get,

$$I(t) = \frac{\sqrt{\pi}\sin\frac{\mu\pi}{2}}{\pi}.\frac{\Gamma(\mu)2^{1-\mu}\sqrt{\pi}}{2^{1-\mu}}t^{-\mu}$$

Thus,

$$I(t) = \sin\frac{\mu\pi}{2}\Gamma(\mu)t^{-\mu}$$

□

Note: In case of $\mu = \frac{1}{2}$,$\mu = -\frac{1}{2}$ we obtain the following integrals,

$$\mu = \frac{1}{2} \Rightarrow \int_0^{+\infty}\frac{\sin tx}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2t}}$$

$$\mu = -\frac{1}{2} \Rightarrow \int_0^{+\infty}\frac{\sin tx}{x\sqrt{x}}\,dx = \sqrt{2t\pi}$$

Corollary 2.1. *we have,*

$$\int_0^{+\infty}\sin\lambda x^p dx = \frac{\Gamma(\frac{1}{p})}{p\lambda^{\frac{1}{p}}}\sin\frac{\pi}{2p} \quad (p>1)$$

Proof. Let us assume that

$$I(\lambda) = \int_0^{+\infty} \sin \lambda x^p \, dx$$

If we set $x^p = u$ in the above relation, after simplifying, one has

$$I(\lambda) = \frac{1}{p} \int_0^{+\infty} u^{\frac{1}{p}-1} \sin \lambda u \, du$$

If we use lemma 2.1, we get

$$I(\lambda) = \frac{1}{p} \int_0^{+\infty} u^{\frac{1}{p}-1} \sin \lambda u \, du = \frac{1}{p} \lambda^{-\frac{1}{p}} \, \Gamma\left(\frac{1}{p}\right) \sin \frac{\pi}{2p}$$

or,

$$I(\lambda) = \frac{\Gamma(\frac{1}{p}) \sin \frac{\pi}{2p}}{p\lambda^{\frac{1}{p}}}$$

□

References

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