Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 22, 1069 - 1075

A Generalization of Lifting Modules

Y. Talebi

Department of Mathematics,
University of Mazandaran,
Babolsar, Iran, 47416-1468

M. J. Nematollahi and Kh. Ghaziani

Islamic Azad University-Arsanjan Branch,
Arsanjan, Iran, 73761-168

Abstract

A module M is called lifting if every submodule N of M, contains a direct summand K of M such that $N/K \ll M/K$. We call a module M cf-lifting if every coessentialy a finitely generated submodule of M lies above a direct summand. In this paper we study about cf-lifting modules and give some conditions for a cf-lifting module to be a lifting module. At the end of this paper we obtain some properties of decomposition of cf-lifting modules.

Mathematics Subject Classification: 16D40, 16D90.

Keywords: lifting module, cf-lifting module, hollow-lifting module, local direct summand, relative projective.

1 Introduction

Extending modules are generalization of injective modules and, dually, *lifting modules* generalize projective supplemented modules. In this paper we generalize the lifting modules as cf-lifting modules.

2 Preliminary Notes

Throughout this paper, R is always an associative ring with unit and $Mod-R$ denotes the category of unital right R-modules. Let M be a module and $S \leq M$. S is called *small* in M (notation $S \ll M$) if $M \neq S+T$ for any proper

submodule T of M. A module M is called *hollow* if every proper submodule of M is small in M. A non-empty family $\{N_\lambda\}_{\lambda\in\Lambda}$ of proper submodules of a module M is called *coindependent* if for every $\lambda \in \Lambda$ and any finite subset $F \subseteq \Lambda \setminus \{\lambda\}$, $N_\lambda + \cap_{i\in F} N_i = M$. A module M is called to have *finite hollow dimension* if M does not contain an infinite coindependent family of submodules. Let N and L be submoduls of M, N is called a *supplement* of L if it is minimal with the property $M = N + L$, equivalently $M = N + L$ and $N \cap L \ll N$. M is *amply supplemented* if for any two submodules A and B of M with $M = A + B$ there exist a supplement P of A such that $P \leq B$. For $A \subseteq B \subseteq M$, A is called a *coessential submodule* of B in M (denoted by $A \leq^{ce} B$) if $B/A \ll M/A$. A submodul A of M is called *coclosed* in M (denoted by $A \leq^{cc} M$) if A has no proper coessential submodule.
M is called *lifting*, if for any submodule N of M, there exist a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$, equivalently M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand of M. We say N *lies above* K in M if $N/K \ll M/K$. An R -module M is called *hollow - lifting* if M is amply supplemented and every hollow submodule of M lies above a direct summand of M. An internal direct sum $\bigoplus_I A_i$ of submodule of a module M is called a *local direct summand* of M if, given any finite subset F of the index set I, the direct sum $\bigoplus_{i\in F} A_i$ is a direct summand of M. The family $\{M_i | i \in I\}$, is called *relatively projective* if M_i is M_j- projective for each $i \neq j; i, j \in I$.
A module$M$ is called to has C_3, provided for any two direct summands M_1 and M_2 of M if $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M. Finally we recall that a ring R is a right *V-ring* if and only if every simple module is injective, if and only if $Rad(M) = 0$ for all R-modules.

Definition 2.1 *Let $N \leq M$, N is called coessentialy finitely generated if it is contained coessentialy in a finitely generated submodule of M (i.e. there exists a finitely generated submodule H of M such that, $N \leq^{ce} H$ in M).*

Definition 2.2 *A module M is called finitely lifting, or f-lifting for short, if every finitely generated submodule of M lies above a direct summand.*

Definition 2.3 *An R-module M is called cf-lifting if every submodule of M which contained coessentialy in a finitely generated submodule lies a bove a direct summand.*

It is clear that every lifting module is cf-lifting.

Definition 2.4 *M is called amply supplemented with respect to coessentialy finitely generated submodules (or c.f.g. amply supplemented for short) in case for each coessentialy finitely generated submodules X and Y such that $M = X + Y$, Y contains a supplement of X.*

Definition 2.5 *Let M_1 and M_2 be modules. The module M_1 is smally M_2-projective if every homomorphism $f : M_1 \to M_2/A$ where A is a submodule of M_2 and $Imf \ll M_2/A$, can be lifted to a homomorphism $\rho : M_1 \to M_2$.*

3 Main Results

Lemma 3.1 *If M is c.f.g. amply supplemented then every submodule of M which contained coessentialy in a finitely generated submodule of M that is not small in M lies above a supplement in M.*

Proof. It is clear by [6, Proposition (2.2)].

Theorem 3.2 *Let M be an R-module. Then the following statements are equivalent:*

1) M is cf-lifting.

2) For every coessentialy finitely generated submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$, $N \cap M_2 \ll M$.

3) Every coessentialy finitely generated submodule N of M can be written as $N = N_1 \oplus N_2$ with N_1 is direct summand of M and $N_2 \ll M$.

4) M is c.f.g amply supplemented and every coessentialy finitely generated coclosed submodule of M is a direct summand of M.

Proof. It is clear By [6, Proposition (1.1)].

Theorem 3.3 *If M is cf-lifting then M is f-lifting.*

Proof. Let A be a finitely generated submodule of M then $A \leq^{ce} A$ in M hence there exists a summand K of M such that $K \leq^{ce} A$ in M then M is f-lifting.

Lemma 3.4 *Let M be an R-module and $K \subseteq L$ be submodules of M. Then if $K \leq^{cc} M$, then $K \leq^{cc} L$ and the converse is true if $L \leq^{cc} M$.*

Proof. By [3, Lemma 2.6]

Theorem 3.5 *Any coclosed submodule (and hence any direct summand) of a cf-lifting module is cf-lifting.*

Proof. Let K be a coclosed submodule of M and M be a cf-lifting module. Let $N \leq^{cc} K$ and $N \leq^{ce} H$ in K such that H is a finitely generated submodule of K. Now $H/N \ll K/N \leq M/N$ implies that $H/N \ll M/N$ i.e. $N \leq^{ce} H$ in M. Since $N \leq^{cc} K$ and $K \leq^{cc} M$, then $N \leq^{cc} M$. So N is a direct summand of M and hence a direct summand of K. It means K is a cf-lifting modules.

Theorem 3.6 *For noetherian R-module M, the following statements are equivalent:*

1)M is cf-lifting.

2)M is f-lifting.

3)M is lifting.

Proof. $1 \Rightarrow 2$ By the above Proposition.
$2 \Rightarrow 3$ Since M is noetherian, every submodule of M is finitely generated so M is lifting.
$3 \Rightarrow 1$ It is trivial.

Lemma 3.7 *Let M be an R-module with finite hollow dimension. Then M is hollow-lifting if and only if M is lifting.*

Proof. By [6, Corollary (1.5)].

Lemma 3.8 *Let M be an hollow-lifting module and $K \leq^{cc} M$ with finite hollow dimension. Then K is a direct summand of M.*

Proof. By [6, Lemma (1.4)].

Theorem 3.9 *Let M be an R-module with finite hollow dimension. If M is hollow-lifting module then M is cf-lifting.*

Proof. It follows immediately from Lemma 1.11 and Lemma 1.12.

Lemma 3.10 *Let $M = M_1 \oplus ... \oplus M_n$ be a finite direct sum of relatively projective modules M_i. Then M is lifting if and only if M is amply supplemented and M_i is lifting for all $1 \leq i \leq n$.*

Proof. By [4, Corollary (2.9)].

Lemma 3.11 *Let $M = M_1 \oplus M_2$ be a weakly supplemented module. Suppose that for every coclosed submodule N of M such that $M = N + M_1$ or $M = N + M_2$, N is a direct summand of M. Let K be a coclosed submodule of M such that every submodule of M/K has an s-closure in M/K. Then K is a direct summand of M.*

Proof. By [4, Proposition (1.8)].

Theorem 3.12 *Let $M = M_1 \oplus M_2$ be an amply supplemented module. Suppose that every coclosed submodule K of M with $M = K + M_1$ is a direct summand of M or every coclosed submodule K of M which is coessentialy finitely generated such that $M = K + M_2$, is a direct summand of M. Then M is cf-lifting.*

Proof. It follows from Lemma 2.1 and Lemma 2.2.

Theorem 3.13 *Let M be an amply supplemented module and $M = M_1 \oplus M_2$ is a direct sum of relatively projective modules M_1 and M_2 such that M_1 is lifting and M_2 is cf-lifting. Then M is cf-lifting.*

Proof. Let N be a coclosed submodule of M such that $M = N + M_1$. There exists a submodule $N^{'}$ of N such that $M = N^{'} + M_1$. Since $M/N^{'}$ is lifting and $N/N^{'}$ is coclosed in $M/N^{'}$, $N/N^{'}$ is direct summand of $M/N^{'}$. Therefore N is direct summand of M. Hence it follows from Proposition 2.3 M is cf-lifting. For the case $M = N + M_2$ it is clear by Proposition 2.3.

Theorem 3.14 *Let M be a module such that every submodule of M has a coclosure in M and M has C_3. If every local direct summand of M is a direct summand , then M is cf-lifting if and only if it is lifting.*

Let $0 \neq K \leq^{cc} M$. For $0 \neq x \in K$, xR has coclosure in M such as A.A is coclosed in M and A is coessential in xR, so A is a direct summand of M ($A \subseteq xR \subset K \subseteq M$). By Zorn's Lemma there exists a maximal local direct summand $N = \oplus_I A_i$ where each $A_i \subset K$. By hypothesis, N is direct summand of M i.e. $M = N \oplus N^{'}$ for some submodule $N^{'}$ of M, so $K = N \oplus (K \cap N^{'})$. Assume that $K \cap N^{'} \neq 0$. Suppose that for $0 \neq x \in N^{'} \cap K$, xR has a coclosure B in M.Hence B is a direct summand of M and $B \cap N = 0$ so $N \oplus B$ is a local direct summand of M, a contradiction with maximality of N. Hence $K \cap N^{'} = 0$ so $K = N$ a direct summand of M.

Theorem 3.15 *A ring R is a V-ring if and only if for any R-module M, every submodule of M is coclosed in M.*

Proof. By [3, Lemma 2.1].

Lemma 3.16 *LetR be a V-ring. An R-module M is lifting if and only if it is semisimple.*

Proof. It is clear by above Proposition.

Corollary 3.17 *Let R be a V-ring and M be a noetherian R-module then M is cf-lifting if and only if it is semisimple.*

Proof. By Proposition (2.6)and Lemma (2.7).

Lemma 3.18 *Let M_1 and M_2 be modules and $M = M_1 \oplus M_2$. The following statements are equivalent:*

1) M_1 is smally M_2-projective.

2) For every submodule N of M such that $(N + M_1/N) \ll M/N$, there exists a submodule N' of N such that $M = N' \oplus M_2$.

Proof. By [4, Lemma (2.4)].

Lemma 3.19 *Let M_1 be any module and M_2 a lifting module and let $M = M_1 \oplus M_2$. If M_1 is smally M_2-projective, then every coclosed submodule N of M such that $(N + M_1)/N \ll M/N$ is a direct summand.*

Proof. By [4, Lemma (2.7)].

Theorem 3.20 *Let M_1 be a lifting module and M_2 be a cf-lifting modules and $M = M_1 \oplus M_2$ be an amply supplemented module. If one of the following conditions holds, then M is cf-lifting.*

1) M_1 is smally M_2-projective and every coclosed submodule N of M such that $M = N + M_1$ is a direct summand.

2) M_1 and M_2 are relatively smally projective and every coclosed submodule N of M such that $M = N + M_1 = N + M_2$ is a direct summand.

3) M_2 is M_1-projective and M_1 is smally M_2-projective .

4) M_1 is semisimple and smally M_2-projective.

Proof. It is clear by [4, Theorem 2.8].

References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules,* Spring Verlag, New York 1974.

[2] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules* , Frontiers in Mathematics, Birkhaeuser Basel 2006.

[3] L. Ganesan and N. Vanaja, Modules for which every Submodule has a Unique Coclouser, *Comm. Algebra,* **30** (2002), 2355-2377.

[4] D. Keskin, On Lifting Modules, *Comm. Algebra,* **28** (2000), 3427-3440.

[5] D. Keskin, On Coclosed Submodules, *Indian J. pure appl. Math.,***36** (2005), 135-144.

[6] C. Lomp, , On Dual Goldie Dimension, *Diplomarbeit (M. Sc. Thesis),Doesseldorf* (1996).

[7] S. H. Mohamed and B. J. Muller, *Continuous and discrite modules,* Londen Math. Soc. LNS 147 Cambridge Univ. press, Cambridge, 1990.

[8] R. Wisbauer, *Foundations of Module and Ring Theory,* Gordon and Breach: Reading, 1991.

Received: March 14, 2007