Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 24, 1163 - 1166

A Note on Common Fixed-Points for Banach Operator Pairs

Yisheng Song

College of Mathematics and Information Science
Henan Normal University, China, 453007
songyisheng123@yahoo.com.cn, songyisheng123@163.com

Sumei Xu

Department of Mathematics and Applied Mathematics
Anyang Normal University, P.R. China

Abstract. In this paper, we obtain some results on Banach operator pair better than those given by J. Chen and Z. Li [Common Fixed-points for Banach Operator Pairs in Best Approximation, J.Math. Anal. Appl. (2007), doi:10.1016/j.jmaa.2007.01.064].

Mathematics Subject Classification: 41A50, 47H10, 54H25

Keywords: $g-$nonexpansive mappings, Common fixed points, Banach operator pairs

Recently, J. Chen and Z. Li[1] introduced the notion of Banach operator pairs as a new class of non-commuting maps, and presented serval common fixed point theorems. They also showed that the concept was of basic importance for the study of common fixed points in best approximation and differ from the classical notion such as the (weakly) compatible maps [2, 3, 6] and $R-$subweakly commuting maps [4, 5].

The ordered pair (T, g) of two self-maps T and g of a metric space E is called *a Banach operator pair*, if the set $F(g)$ of fixed-points of g is $T-$invariant, namely $T(F(g)) \subset F(g)$. The following theorem is one of their main results in [1].

Theorem CL([1, Theorem 3.2]) *Let S be a weakly compact subset of a normed space E which is starshaped with respect to $p \in S$, and let T and g are two self-maps of S such that (T, g) is a Banach operator pair on S, T is $g-$nonexpansive on S, and $p \in F(g)$. If g is both weakly continuous* *and strongly continuous on* S*,* $F(g)$ *is starshaped with respect to* p*,* $Cl(T(S))$ *is complete, and if either (i)* E *satisfies Opial's condition, or (ii)* $g - T$ *is demiclosed on* S, *then* $F(T, g) \neq \emptyset$*.*

After carefully read their works, we observed the conditions of Theorem CL is quite rigid also. We attempt to simplify their proof so that remove some conditions. Fortunately, the desired conclusion is reached which is the following.

Theorem 1 *Let* K *be a weakly compact subset of a normed space* E *which is starshaped with respect to* $p \in K$*, and let* f *and* g *are two self-maps of* K *such that* (T, g) *is a Banach operator pair on* K*,* T *is* $g-$*nonexpansive on* K*,* $F(g)$ *is starshaped with respect to* p*,* $Cl(T(K))$ *is complete, and* $p \in F(g)$*. Then* $F(T, g) \neq \emptyset$ *if one of the following conditions holds:*
(i) g *is strongly continuous on* K*;*
(ii) g *is weakly continuous on* K *and* E *satisfies Opial's condition;*
(iii) g *is weakly continuous on* K *and* $I - T$ *is demiclosed at* 0*;*
(iv) both T *and* g *are weakly continuous on* K*.*

Subsequently, we give some necessary concepts in our proof process. Let K be a nonempty subset of a linear normed space E, T and g be two selfmaps of K, and $F(T)$ and $F(T, g)$ denote the set of fixed points of T and the set of common fixed points of T, g, respectively. When $\{x_n\}$ is a sequence in E, then $x_n \to x$(respectively, $x_n \rightharpoonup x$) will stand for strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x.

A normed space E is said to satisfy *Opial's condition* if for every sequence $\{x_n\} \subset E$ weakly convergent to $x \in E$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n_i\to\infty} \|x_n - y\|$$

holds for all $y \neq x$. The set K is called $q-$*starshaped* with $q \in K$ if $kx + (1 - k)q \in K$ for all $x \in K$ and all $k \in [0, 1]$. The selfmap T on K is called $g-$*nonexpansive* if $\|Tx - Ty\| \leq \|g(x) - g(y)\|$ for any $x, y \in K$. If $g = I$, an identity operator, then T is called nonexpansive. A mapping $T : K \to K$ is called *demiclosed* at 0 if for every sequence $\{x_n\} \subset K$ such that $x_n \rightharpoonup x$ and $Tx_n \to 0$ implies $Tx = 0$.

A mapping $T : K \to K$ is called *continuous* if for all $\{x_n\} \subset K$ such that $x_n \to x$ implies that $Tx_n \to Tx$; *strongly continuous* if for all $\{x_n\} \subset K$ such that $x_n \rightharpoonup x$ implies that $Tx_n \to Tx$; *weakly continuous* if for all $\{x_n\} \subset K$ such that $x_n \rightharpoonup x$ implies that $Tx_n \rightharpoonup Tx$.

Lemma 1([1, Lemma 3.1] *Suppose* T *and* g *are two self-maps of a closed subset* K *of the metric space* E *with the metric* d*, such that* (T, g) *is a Banach operator pair on* K *and* T *is* $g-$*contractive on* K*, i.e.* $d(Tx, Ty)!Ukd(g(x), g(y))$*, for all* $x, y \in K$*, with fixed* $k \in [0, 1)$*. If* $F(g)$ *is non-empty and* $Cl(T(K))$ *is complete, then* $F(T, g)$ *is a singleton.*

Next, we present the proof of Theorem 1.

Proof. Let $\{k_n\}$ be a sequence of real numbers such that $0 < k_n < 1$ and $k_n \to 1$ as $n \to \infty$. Define a sequence T_n of maps on K by $T_n x = k_n Tx + (1 - k_n)p$ for all $x \in K$. Then, for each n, the map T_n does carry K into itself since K is $p-$starshaped. It follows from the $g-$nonexpansivity of T that T_n is $g-$contractive on K for each n since for all $x, y \in K$,

$$\|T_n x - T_n y\| = \|k_n Tx - k_n Ty\| \leq k_n \|g(x) - g(y)\|.$$

Furthermore, for each n, (T_n, g) is a Banach operator pair on K. Indeed, the fact that $x \in F(g)$ implies $Tx \in F(g)$, and hence $T_n x = k_n Tx + (1 - k_n)p \in F(g)$ because $F(g)$ is $p-$starshaped. Now by Lemma 2, for each n, there exists $x_n \in K$ such that

$$x_n = g(x_n) = k_n Tx_n + (1 - k_n)p. \tag{1}$$

The weak compactness of K implies that there exists $\{x_{n_i}\} \subset \{x_n\}$ such that

$$x_{n_i} = g(x_{n_i}) = k_{n_i} Tx_{n_i} + (1 - k_{n_i})p \rightharpoonup z \in K. \tag{2}$$

(i) It follows from the strong continuity of g and (2) that $g(x_{n_i}) \to g(z)$. Then $g(z) = z$ since $z \leftharpoonup x_{n_i} = g(x_{n_i}) \rightharpoonup g(z)$.

From (1), we also have

$$\begin{aligned}\|x_{n_i} - Tz\| \leq & k_{n_i} \|Tx_{n_i} - Tz\| + (1 - k_{n_i})\|p - f(z)\| \\ \leq & k_{n_i} \|g(x_{n_i}) - g(z)\| + (1 - k_{n_i})\|p - Tz\| \to 0\end{aligned}$$

since $g(x_{n_i}) \to g(z)$ and $k_{n_i} \to 1$. By the weakly lower semicontinuity of the norm, we have

$$\|z - Tz\| \leq \liminf_{n_i \to \infty} \|x_{n_i} - Tz\| = 0.$$

Then $z = Tz = g(z)$, and so $z \in F(T, g)$.

(ii) The weak continuity of g and (2) imply that $g(x_{n_i}) \rightharpoonup g(z)$. Then $g(z) = z$. It follows from the weak compactness of K that $\{x_n\}$ and $\{g(x_n)\}$ are bounded, and so is $\{Tx_n\}$ because $\|Tx_n - Tp\| \leq \|g(x_n) - g(p)\|$. Therefore,

$$\lim_{n \to \infty} \|x_n - Tx_n\| = \lim_{n \to \infty} (1 - k_n)\|p - Tx_n\| = 0. \tag{3}$$

Suppose that $z \neq Tz$. Since X satisfies Opial's condition, then

$$\begin{aligned}\liminf_{i \to \infty} \|x_{n_i} - z\| <& \liminf_{i \to \infty} \|x_{n_i} - Tz\| \\ \leq& \liminf_{i \to \infty} (\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tz\|) \\ \leq& \liminf_{i \to \infty} \|g(x_{n_i}) - g(z)\| = \liminf_{i \to \infty} \|x_{n_i} - z\|,\end{aligned}$$

which is a contradiction. Therefore $z = Tz$ and so $z \in F(T, g)$.

(iii) Similarly to (ii), we obtain that

$$x_{n_i} \rightharpoonup z = g(z) \text{ and } \lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$

By the fact that $I - T$ is demiclosed at 0, we get $z - Tz = 0$. Hence, $z \in F(T, g)$.

(iv) It follows from the fact that $x_{n_i} \rightharpoonup z$ and the weak continuity of T and g that $Tx_{n_i} \rightharpoonup Tz$ and $g(x_{n_i}) \rightharpoonup g(z)$, respectively. Thus, noticing $k_n \to 1$,

$x_{n_i} = g(x_{n_i}) = k_{n_i} T x_{n_i} + (1 - k_{n_i})p \rightharpoonup Tz$. Hence, we have $z = Tz = g(z)$. This complete the proof. □

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Received: April 6, 2007