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The Recurrent Reimannian Spaces Having a Semi-symmetric Metric Connection and a Decomposable Curvature Tensor

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Abstract

In this paper, we have studied the recurrent Riemannian space having a semi-symmetric metric connection and the curvature tensor of which is decomposed in the form $R^i_{jkl} = v^i \varphi_{jkl}$ and proved some theorems concerning such spaces.

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1 Introduction

Suppose a Riemannian manifold V_n of dimension n with metric g_{ij} admits a metric semi-symmetric connection ∇ with connection coefficients Γ^h_{ji} given by [4,1]

$$\Gamma^h_{ji} = T^h_{ji} + \delta^h_j p_i - g_{ji} p^h \ , \tag{1}$$

where T^h_{ji} are Christoffel symbols of V_n, p_i is a arbitrary gradient vector field and $p^h = g^{ha} p_a$.

The components of the mixed curvature tensor and the Ricci tensor of V_n are respectively

$$R^h_{kji} = \partial_k \Gamma^h_{ji} - \partial_j \Gamma^h_{ki} + \Gamma^h_{km} \Gamma^m_{ji} - \Gamma^h_{jm} \Gamma^m_{ki} \tag{2}$$

$$R_{ji} = R^h_{hji} \ . \tag{3}$$

The curvature tensor of V_n satisfies the following relations [2]

$$R^h_{kji} + R^h_{jki} = 0 \tag{4}$$

$$R^h_{kji} + R^h_{jik} + R^h_{ikj} = 0. \tag{5}$$

If p is a arbitrary scalar function defined in V_n then the relation

$$\nabla_l(e^{-2p}R^h_{kji}) + \nabla_k(e^{-2p}R^h_{jli}) + \nabla_j(e^{-2p}R^h_{lki}) = 0 \tag{6}$$

holds true. [2]

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If the curvature tensor R^i_{jkl} of V_n satisfies the condition

$$\nabla_m R^h_{kji} = \psi_m R^h_{kji}, \tag{7}$$

where ψ_m is a covariant vector field, then V_n is called recurrent. [3]

In this section we consider the recurrent Riemannian spaces denoted by RV_n the curvature tensor of which is decomposed in the form

$$R^h_{kji} = v^h \varphi_{kji}, \tag{8}$$

where v^h is a contravariant vector field and φ_{kji} is a covariant tensor field.

Using the relations (4) and (8) we get

$$\varphi_{kji} + \varphi_{jki} = 0. \tag{9}$$

Taking into account the relations (6), (7) and (8) we obtain

$$(\psi_l - 2p_l)\varphi_{kji} + (\psi_k - 2p_k)\varphi_{jli} + (\psi_j - 2p_j)\varphi_{lki} = 0. \tag{10}$$

Multiplying both hand sides of (10) by v^l and summing for l we have

$$\varphi_{kji} = \alpha[(\psi_k - 2p_k)\phi_{ji} - (\psi_j - 2p_j)\phi_{ki}], \tag{11}$$

where α is a scalar function which is defined by

$$\alpha = \frac{1}{\mu} \quad (\mu = (\psi_l - 2p_l)v^l) \tag{12}$$

and ϕ_{ki} is a covariant tensor field which is defined by

$$\phi_{ki} = v^l \varphi_{lki}. \tag{13}$$

Using the relations (3), (8) and (13) we obtain

$$R_{ji} = \phi_{ji}. \tag{14}$$

Using (14) in (11) we get

$$\varphi_{kji} = \alpha[(\psi_k - 2p_k)R_{ji} - (\psi_j - 2p_j)R_{ki}]. \tag{15}$$

Multiplying the relation (9) by $v^k v^j$ and summing for k and j and using the relations (13) and (14) we obtain

$$R_{ji}v^j = 0 \tag{16}$$

from which it follows that

$$det(R_{ji}) = 0. \tag{17}$$

Thus we obtain the

Theorem 2.1 *If a space RV_n has a decomposable curvature tensor in the form $R^h_{kji} = v^h \varphi_{kji}$, then we have*

$$\varphi_{kji} = \alpha[(\psi_k - 2p_k)R_{ji} - (\psi_j - 2p_j)R_{ki}]$$

and

$$det(R_{ji}) = 0.$$

Theorem 2.2 *If a space RV_n has a decomposable curvature tensor in the form $R^h_{kji} = v^h \varphi_{kji}$, then the vector field v^h and the tensor field φ_{kji} are recurrent.*

Proof: By (3) and (7) we get

$$\nabla_m R_{ji} = \psi_m R_{ji}. \tag{18}$$

On the other hand, from (8) and (15) we can write the curvature tensor R^h_{kji} of RV_n in the form

$$R^h_{kji} = \alpha v^h[(\psi_k - 2p_k)R_{ji} - (\psi_j - 2p_j)R_{ki}]. \tag{19}$$

Taking the covariant derivative both hand sides of (19) with respect to the coordinate x^m and using the relations (7) and (18) we obtain

$$R^a_{kji}\nabla_m v^h = R^h_{kji}\nabla_m v^a. \tag{20}$$

If we use the relation (8) then the relation (20) reduces to

$$v^a \nabla_m v^h = v^h \nabla_m v^a \tag{21}$$

from which it follows that

$$\nabla_m v^h = \lambda_m v^h, \tag{22}$$

where λ_m is a covariant vector field. From the relation (22) we have the vector field v^h is recurrent.

On the other hand taking the covariant derivative of (8) with respect to the coordinate x^m and using the relations (7) and (22) we get

$$\nabla_m \varphi_{kji} = (\psi_m - \lambda_m)\varphi_{kji}. \tag{23}$$

From the relation (23) we have the tensor field φ_{kji} is recurrent.

Now we consider the RV_n spaces having a decomposable curvature tensor in the form

$$R^h_{kji} = v^h(\psi_i - 2p_i)\varphi_{kj}, \quad \varphi_{kji} = (\psi_i - 2p_i)\varphi_{kj}, \tag{24}$$

where φ_{kj} is a covariant tensor field and concerning these spaces we prove the following theorem:

Theorem 2.3 *The tensor field φ_{kji} may be decomposed in the form $\varphi_{kji} = (\psi_i - 2p_i)\varphi_{kj}$ if and only if the condition*

$$\nabla_m(\psi_i - 2p_i) + (\lambda_m - \alpha\nabla_m\mu)(\psi_i - 2p_i) = 0$$

holds true.

Proof: We first prove the necessity of the condition. Suppose that the tensor φ_{kji} is decomposed in the form (24). Under this condition the equation (23) transforms into

$$(\psi_m - \lambda_m)(\psi_i - 2p_i)\varphi_{kj} = \nabla_m(\psi_i - 2p_i)\varphi_{kj} + (\psi_i - 2p_i)(\nabla_m\varphi_{kj}). \tag{25}$$

Taking the covariant derivative of the equation $\mu = (\psi_l - 2p_l)$ with respect to x^m and using (22) we obtain

$$v^i\nabla_m(\psi_i - 2p_i) = \nabla_m\mu - \lambda_m\mu. \tag{26}$$

Multiplying both sides (25) by v^i and summing for i and taking (26) into account, we get

$$\nabla_m\varphi_{kj} = (\psi_m - \alpha\nabla_m\mu)\varphi_{kj}. \tag{27}$$

Taking the covariant derivative of both sides of the equation $R^h_{kji} = v^h(\psi_i - 2p_i)\varphi_{kj}$ with respect to x^m and using (7), (22) and (27) we obtain

$$\psi_m R^h_{kji} = \psi_m R^h_{kji} + [\nabla_m(\psi_i - 2p_i) + (\lambda_m - \alpha\nabla_m\mu)(\psi_i - 2p_i)]\varphi_{kj}v^h \tag{28}$$

from which it follows that

$$\nabla_m(\psi_i - 2p_i) + (\lambda_m - \alpha\nabla_m\mu)(\psi_i - 2p_i) = 0. \tag{29}$$

Conversely, the condition (29) is sufficient. To see this, take the covariant derivative of both sides of (29) with respect to x^n we get

$$\nabla_n\nabla_m(\psi_i - 2p_i) + \nabla_n[(\lambda_m - \alpha\nabla_m\mu)(\psi_i - 2p_i)] = 0. \tag{30}$$

Interchanging the indices m and n in (30) and subtracting the equation so obtained from (30) we obtain

$$\varphi_{nmi} = \alpha[p_m(\lambda_n - \alpha\nabla_n\mu) - p_n(\lambda_m - \alpha\nabla_m\mu)$$

$$+\nabla_m(\lambda_n - \alpha\nabla_n\mu) - \nabla_n(\lambda_m - \alpha\nabla_m\mu)](\psi_i - 2p_i), \tag{31}$$

where we have used the relations (8), (29) and the Ricci identity.

From (31) it follows that the tensor φ_{nmi} may be written in the form

$$\varphi_{nmi} = (\psi_i - 2p_i)\varphi_{nm} \quad ,$$

where φ_{nm} is given by

$$\varphi_{nm} = \alpha[p_m(\lambda_n - \alpha\nabla_n\mu) - p_n(\lambda_m - \alpha\nabla_m\mu)$$

$$+\nabla_m(\lambda_n - \alpha\nabla_n\mu) - \nabla_n(\lambda_m - \alpha\nabla_m\mu)]. \tag{32}$$

The proof of the Theorem is completed.

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