Some Results about the Classification of
Totally Real Minimal Surfaces in $S^5$

Rodrigo Ristow Montes

Departamento de Matemática
Universidade Federal da Paraíba
BR– 58.051-900 João Pessoa, P.B., Brazil
ristow@mat.ufpb.br

Abstract

In this paper we prove that the Contact angle ($\pi/4 \leq \beta < \pi/2$) for
totally real compact minimal surfaces in the sphere $S^5$ with a parallel	normal vector field must be constant.

Mathematics Subject Classification: 53C42 - 53C40 - 53C21

Keywords: contact angle, holomorphic angle, minimal surfaces, parallel
field, totally real surfaces.

1 Introduction

In [4] we introduced the notion of contact angle, that can be considered as a new
generic invariant useful to investigate the geometry of immersed surfaces in
$S^3$. Geometrically, the contact angle ($\beta$) is the complementary angle between
the contact distribution and the tangent space of the surface. Also in [4], we
deduced formulas for the Gaussian curvature and the Laplacian of an immersed
minimal surface in $S^3$, and we gave a characterization of the Clifford Torus as
the only minimal surface in $S^3$ with constant contact angle.

We define $\alpha$ to be the angle given by $\cos \alpha = \langle ie_1, v \rangle$, where $e_1$ and $v$
are defined in section 2. The holomorphic angle $\alpha$ is the analogue of the Kähler
angle introduced by Chern and Wolfson in [2].

Recently, in [5], we construct a family of minimal tori in $S^5$ with constant
contact and holomorphic angle. These tori are parametrized by the following
circle equation

$$a^2 + \left(b - \frac{\cos \beta}{1 + \sin^2 \beta}\right)^2 = 2 \frac{\sin^4 \beta}{(1 + \sin^2 \beta)^2},$$

(1)
where $a$ and $b$ are given in Section 3 (equation (9)). In particular, when $a = 0$ in (1), we recover the examples found by Kenmotsu, in [3]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$ in (1), we find a new family of minimal tori in $S^5$, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. Also, in [5], when $\beta = \frac{\pi}{2}$, we give an alternative proof of this classification of a Theorem from Blair in [1], and Yamaguchi, Kon and Miyahara in [6] for Legendrian minimal surfaces in $S^5$ with constant Gaussian curvature.

In this paper, we will classify totally real (see section 4) compact minimal surfaces in $S^5$ with a parallel normal vector field. We suppose that $e_3$ (in equation (3)) is a parallel normal vector field, and we get the following

**Theorem 1.1.** The contact angle $\pi/4 \leq \beta < \pi/2$ is constant for totally real compact minimal surfaces in $S^5$ with null principal curvatures $a, b$.

## 2 Contact Angle for Immersed Surfaces in $S^{2n+1}$

Consider in $\mathbb{C}^{n+1}$ the following objects:

- the Hermitian product: $(z, w) = \sum_{j=0}^{n} z^j \bar{w}^j$;
- the inner product: $\langle z, w \rangle = Re(z, w)$;
- the unit sphere: $S^{2n+1} = \{ z \in \mathbb{C}^{n+1} | (z, z) = 1 \}$;
- the Reeb vector field in $S^{2n+1}$, given by: $\xi(z) = iz$;
- the contact distribution in $S^{2n+1}$, which is orthogonal to $\xi$:
  \[ \Delta_z = \{ v \in T_z S^{2n+1} | \langle \xi, v \rangle = 0 \}. \]

We observe that $\Delta$ is invariant by the complex structure of $\mathbb{C}^{n+1}$.

Let now $S$ be an immersed orientable surface in $S^{2n+1}$.

**Definition 2.1.** The contact angle $\beta$ is the complementary angle between the contact distribution $\Delta$ and the tangent space $TS$ of the surface.

Let $(e_1, e_2)$ be a local frame of $TS$, where $e_1 \in TS \cap \Delta$. Then $cos \beta = \langle \xi, e_2 \rangle$. Finally, let $v$ be the unit vector in the direction of the orthogonal projection of $e_2$ on $\Delta$, defined by the following relation

$$e_2 = \sin \beta v + \cos \beta \xi. \quad (2)$$
3 Equations for Gaussian curvature and Laplacian of a minimal surface in $S^5$

In this section, we deduce the equations for the Gaussian curvature and for the Laplacian of a minimal surface in $S^5$ in terms of the contact angle and the holomorphic angle. Consider the normal vector fields

$$
e_3 = i \csc \alpha e_1 - \cot \alpha v$$
$$e_4 = \cot \alpha e_1 + i \csc \alpha v$$
$$e_5 = \csc \beta \xi - \cot \beta e_2$$

where $\beta \neq 0, \pi$ and $\alpha \neq 0, \pi$. We will call $(e_j)_{1 \leq j \leq 5}$ an adapted frame.

Using (2) and (3), we get

$$v = \sin \beta e_2 - \cos \beta e_5, \quad iv = \sin \alpha e_4 - \cos \alpha e_1$$
$$\xi = \cos \beta e_2 + \sin \beta e_5$$

It follows from (3) and (4) that

$$ie_1 = \cos \alpha \sin \beta e_2 + \sin \alpha e_3 - \cos \alpha \cos \beta e_5$$
$$ie_2 = - \cos \beta z - \cos \alpha \sin \beta e_1 + \sin \alpha \sin \beta e_4$$

Consider now the dual basis $(\theta^j)$ of $(e_j)$. The connection forms $(\theta^j_k)$ are given by

$$De_j = \theta^j_k e_k,$$

and the second fundamental form with respect to this frame are given by

$$\text{II}^j = \theta^j_1 \theta^1 + \theta^j_2 \theta^2; \quad j = 3, ..., 5.$$ \hspace{1cm}

Using (5) and differentiating $v$ and $\xi$ on the surface $S$, we get

$$D\xi = - \cos \alpha \sin \beta \theta^2 e_1 + \cos \alpha \sin \beta \theta^1 e_2 + \sin \alpha \theta^1 e_3 + \sin \alpha \sin \beta \theta^2 e_4$$
$$- \cos \alpha \cos \beta \theta^1 e_5,$$
$$Dv = (\sin \beta \theta^1 - \cos \beta \theta^2) e_1 + \cos \beta (d\beta - \theta^2_5) e_2 + (\sin \beta \theta^2_2 - \cos \beta \theta^3_5) e_3$$
$$+ (\sin \beta \theta^2 - \cos \beta \theta^4_5) e_4 + \sin \beta (d\beta + \theta^5_2) e_5.$$
Differentiating $e_3$, $e_4$ and $e_5$, we have

\[
\begin{align*}
\theta^1_3 &= -\theta^2_1 \\
\theta^2_3 &= \sin \beta (d\alpha + \theta^1_4) - \cos \beta \sin \alpha \theta^1 \\
\theta^3_3 &= \csc \beta \theta^2_1 - \cot \alpha (\theta^2_1 + \csc \beta \theta^1_4) \\
\theta^4_3 &= \cot \beta \theta^2_1 - \csc \beta \sin \alpha \theta^1 \\
\theta^1_4 &= -d\alpha - \csc \beta \theta^3_1 + \sin \alpha \cot \beta \theta^1 \\
\theta^2_4 &= -\theta^1_2 \\
\theta^3_4 &= \csc \beta \theta^2_1 + \cot \alpha (\theta^2_1 + \csc \beta \theta^1_4) \\
\theta^4_4 &= \cot \beta \theta^2_1 - \sin \alpha \theta^2 \\
\theta^5_4 &= -\cos \alpha \theta^2 - \cot \beta \theta^1_2 \\
\theta^2_5 &= d\beta + \cos \alpha \theta^1 \\
\theta^3_5 &= -\cot \beta \theta^2_1 + \csc \beta \sin \alpha \theta^1 \\
\theta^4_5 &= -\cot \beta \theta^2_1 + \sin \alpha \theta^2.
\end{align*}
\]

The conditions of minimality and of symmetry are equivalent to the following equations:

\[
\theta^3_1 \wedge \theta^1 + \theta^3_2 \wedge \theta^2 = 0 = \theta^3_1 \wedge \theta^2 - \theta^3_2 \wedge \theta^1.
\] (8)

On the surface $S$, we consider

\[
\theta^3_1 = a \theta^1 + b \theta^2
\]

It follows from (8) that

\[
\begin{align*}
\theta^3_1 &= a \theta^1 + b \theta^2 \\
\theta^3_2 &= b \theta^1 - a \theta^2 \\
\theta^1_4 &= d\alpha + (b \csc \beta - \sin \alpha \cot \beta) \theta^1 - a \csc \beta \theta^2 \\
\theta^2_4 &= d\alpha \circ J - a \csc \beta \theta^1 - (b \csc \beta - \sin \alpha \cot \beta) \theta^2 \\
\theta^5_4 &= d\beta \circ J - \cos \alpha \theta^2 \\
\theta^1_5 &= -d\beta - \cos \alpha \theta^1
\end{align*}
\]

where $J$ is the complex structure of $S$ is given by $Je_1 = e_2$ and $Je_2 = -e_1$.

Moreover, the normal connection forms are given by:

\[
\begin{align*}
\theta^4_3 &= -\sec \beta d\beta \circ J - \cot \alpha \csc \beta \circ J + a \cot \alpha \cot^2 \beta \theta^1 \\
&\quad + (b \cot \alpha \cot^2 \beta - \cos \alpha \cot \beta \csc \beta + 2 \sec \beta \cos \alpha) \theta^2 \\
\theta^5_3 &= (b \cot \beta - \csc \beta \sin \alpha) \theta^1 - a \cot \beta \theta^2 \\
\theta^5_4 &= \cot \beta (d\alpha \circ J) - a \cot \beta \csc \beta \theta^1 + (-b \csc \beta \cot \beta + \sin \alpha (\cot^2 \beta - 1)) \theta^2.
\end{align*}
\]
while the Gauss equation is equivalent to the equation:

\[ d\theta^1_2 + \theta^1_k \wedge \theta^k_2 = \theta^1 \wedge \theta^2. \] (11)

Therefore, using equations (9) and (11), we have

\[
K = 1 - |\nabla \beta|^2 - 2 \cos \alpha \beta_1 - \cos^2 \alpha - (1 + \csc^2 \beta)(a^2 + b^2) + 2b \sin \alpha \csc \beta \cot \beta + 2 \sin \alpha \cot \beta \alpha_1 - |\nabla \alpha|^2 \\
+ 2a \csc \beta \alpha_2 - 2b \csc \beta \alpha_1 - \sin^2 \alpha \cot^2 \beta \\
= 1 - (1 + \csc^2 \beta)(a^2 + b^2) - 2b \csc \beta(\alpha_1 - \sin \alpha \cot \beta) + 2a \csc \beta \alpha_2 \\
- |\nabla \beta + \cos \alpha e_1|^2 - |\nabla \alpha - \sin \alpha \cot \beta e_1|^2 
\] (12)

Using (7) and the complex structure of \( S^5 \), we get

\[
\theta^1_2 = \tan \beta(d\beta \circ J - 2 \cos \alpha \theta^2) 
\] (13)

Differentiating (13), we conclude that

\[
d\theta^1_2 = (-1 + \tan^2 \beta)|\nabla \beta|^2 - \tan \beta \Delta \beta - 2 \cos \alpha(1 + 2 \tan^2 \beta)\beta_1 + 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos^2 \alpha)\theta^1 \wedge \theta^2 
\]

where \( \Delta = tr \nabla^2 \) is the Laplacian of \( S \). The Gaussian curvature is therefore given by:

\[
K = -1 + \tan^2 \beta)|\nabla \beta|^2 - \tan \beta \Delta \beta - 2 \cos \alpha(1 + 2 \tan^2 \beta)\beta_1 + 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos^2 \alpha. 
\] (14)

From (12) and (14), we obtain the following formula for the Laplacian of \( S \):

\[
\tan \beta \Delta \beta = (1 + \csc^2 \beta)(a^2 + b^2) + 2b \csc \beta(\alpha_1 - \sin \alpha \cot \beta) - 2a \csc \beta \alpha_2 \\
-\tan^2 \beta(|\nabla \beta + 2 \cos \alpha e_1|^2 - |\cot \beta \nabla \alpha + \sin \alpha(1 - \cot^2 \beta)e_1|^2) \\
+ \sin^2 \alpha(1 - \tan^2 \beta) 
\] (15)

4 Gauss-Codazzi-Ricci equations for totally real minimal surfaces in \( S^5 \) with null principal curvatures

**Definition 4.1.** We define a totally real minimal surface as a minimal surface in \( S^5 \) with Holomorphic angle \( \alpha = \pi/2 \).
Using the connection form (9), (10) and (13) with $\alpha = \pi/2$ and $a, b$ nulls, we have

\begin{align*}
\theta_1^3 &= 0 = \theta_2^3 \\
\theta_1^4 &= -\cot \beta \theta_1^1 \\
\theta_2^2 &= \cot \beta \theta_2^2 \\
\theta_1^5 &= \beta_2 \theta_1^1 - \beta_1 \theta_2^1 \\
\theta_2^5 &= -\beta_1 \theta_1^1 - \beta_2 \theta_2^2 \\
\theta_1^5 &= \beta_2 \theta_1^1 - \beta_1 \theta_2^2 \\
\theta_3^4 &= -\sec \beta (\beta_2 \theta_1^1 - \beta_1 \theta_2^2) \\
\theta_3^5 &= -\csc \beta \theta_1^1 \\
\theta_5^4 &= \beta_2 \theta_1^1 - \beta_1 \theta_2^2 \\
\theta_5^1 &= \tan \beta (\beta_2 \theta_1^1 - \beta_1 \theta_2^2)
\end{align*}

(16)

Now Codazzi-Ricci equations:

\begin{align*}
d\theta_3^2 + \theta_3^1 \wedge \theta_1^2 + \theta_3^4 \wedge \theta_4^1 + \theta_3^5 \wedge \theta_5^1 &= 0 \\
d\theta_4^2 + \theta_4^1 \wedge \theta_1^2 + \theta_4^3 \wedge \theta_3^1 + \theta_4^5 \wedge \theta_5^1 &= 0 \\
d\theta_5^2 + \theta_5^1 \wedge \theta_1^2 + \theta_5^3 \wedge \theta_3^1 + \theta_5^4 \wedge \theta_4^1 &= 0
\end{align*}

simplify to:

\[2 \csc \beta \beta_1 = 0\]

(17)

Therefore:

\[\beta_1 = 0\]

(18)

The following Codazzi-Ricci equations are always verified:

\begin{align*}
d\theta_3^2 + \theta_3^1 \wedge \theta_1^2 + \theta_3^4 \wedge \theta_4^1 + \theta_3^5 \wedge \theta_5^1 &= 0 \\
d\theta_4^2 + \theta_4^1 \wedge \theta_1^2 + \theta_4^3 \wedge \theta_3^1 + \theta_4^5 \wedge \theta_5^1 &= 0 \\
d\theta_5^2 + \theta_5^1 \wedge \theta_1^2 + \theta_5^3 \wedge \theta_3^1 + \theta_5^4 \wedge \theta_4^1 &= 0
\end{align*}

\begin{align*}
d\theta_3^3 + \theta_3^1 \wedge \theta_1^3 + \theta_3^4 \wedge \theta_4^3 + \theta_3^5 \wedge \theta_5^3 &= 0 \\
d\theta_4^3 + \theta_4^1 \wedge \theta_1^3 + \theta_4^5 \wedge \theta_5^3 &= 0 \\
d\theta_5^3 + \theta_5^1 \wedge \theta_1^3 + \theta_5^2 \wedge \theta_2^3 + \theta_5^4 \wedge \theta_4^3 + \theta_5^5 \wedge \theta_5^3 &= 0
\end{align*}

\[\Delta (\beta) = -\tan^2 \beta |\nabla \beta|^2 + \cot^2 \beta - 1\]

(19)
5 Proof of the Theorem 1.1

In this section, we will give a prove of the theorem, using Gauss-Codazzi-Ricci equations for a compact minimal surface in $S^5$ with constant holomorphic angle $(\alpha = \pi/2)$ and null principal curvatures $a, b$. Now suppose that $(\pi/4 \leq \beta < \pi/2)$ at the equation (19), then $\Delta \beta < 0$ and using the Hopf’s Lemma, we get the Theorem (1.1).

References


Received: April 6, 2007