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Some Results about the Classification of Totally Real Minimal Surfaces in S^5

Rodrigo Ristow Montes

Departamento de Matemática
Universidade Federal da Paraíba
BR– 58.051-900 João Pessoa, P.B., Brazil
ristow@mat.ufpb.br

Abstract

In this paper we prove that the Contact angle ($\pi/4 \leq \beta < \pi/2$) for totally real compact minimal surfaces in the sphere S^5 with a parallel normal vector field must be constant.

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1 Introduction

In [4] we introduced the notion of contact angle, that can be considered as a new geometric invariant useful to investigate the geometry of immersed surfaces in S^3. Geometrically, the contact angle (β) is the complementary angle between the contact distribution and the tangent space of the surface. Also in [4], we deduced formulas for the Gaussian curvature and the Laplacian of an immersed minimal surface in S^3, and we gave a characterization of the Clifford Torus as the only minimal surface in S^3 with constant contact angle.
We define α to be the angle given by $\cos\alpha = \langle ie_1, v\rangle$, where e_1 and v are defined in section 2. The holomorphic angle α is the analogue of the Kähler angle introduced by Chern and Wolfson in [2].
Recently, in [5], we construct a family of minimal tori in S^5 with constant contact and holomorphic angle. These tori are parametrized by the following circle equation

$$a^2 + \left(b - \frac{\cos\beta}{1+\sin^2\beta}\right)^2 = 2\frac{\sin^4\beta}{(1+\sin^2\beta)^2}, \tag{1}$$

where a and b are given in Section 3 (equation (9)). In particular, when $a = 0$ in (1), we recover the examples found by Kenmotsu, in [3]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$ in (1), we find a new family of minimal tori in S^5, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. Also, in [5], when $\beta = \frac{\pi}{2}$, we give an alternative proof of this classification of a Theorem from Blair in [1], and Yamaguchi, Kon and Miyahara in [6] for Legendrian minimal surfaces in S^5 with constant Gaussian curvature.
In this paper, we will classify totally real (see section 4) compact minimal surfaces in S^5 with a parallel normal vector field. We suppose that e_3 (in equation (3)) is a parallel normal vector field, and we get the following

Theorem 1.1. *The Contact angle* $(\pi/4 \leq \beta < \pi/2)$ *is constant for totally real compact minimal surfaces in* S^5 *with null principal curvatures* a, b

2 Contact Angle for Immersed Surfaces in S^{2n+1}

Consider in $\mathbb{C}^{n+1}$ the following objects:

- the Hermitian product: $(z, w) = \sum_{j=0}^{n} z^j \bar{w}^j$;
- the inner product: $\langle z, w\rangle = Re(z, w)$;
- the unit sphere: $S^{2n+1} = \{z \in \mathbb{C}^{n+1} | (z, z) = 1\}$;
- the *Reeb* vector field in S^{2n+1}, given by: $\xi(z) = iz$;
- the contact distribution in S^{2n+1}, which is orthogonal to ξ:

$$\Delta_z = \{v \in T_z S^{2n+1} | \langle \xi, v\rangle = 0\}.$$

We observe that Δ is invariant by the complex structure of $\mathbb{C}^{n+1}$.

Let now S be an immersed orientable surface in S^{2n+1}.

Definition 2.1. *The* contact angle β *is the complementary angle between the contact distribution* Δ *and the tangent space* TS *of the surface.*

Let (e_1, e_2) be a local frame of TS, where $e_1 \in TS \cap \Delta$. Then $\cos\beta = \langle \xi, e_2\rangle$. Finally, let v be the unit vector in the direction of the orthogonal projection of e_2 on Δ, defined by the following relation

$$e_2 = \sin\beta v + \cos\beta \xi. \tag{2}$$

3 Equations for Gaussian curvature and Laplacian of a minimal surface in S^5

In this section, we deduce the equations for the Gaussian curvature and for the Laplacian of a minimal surface in S^5 in terms of the contact angle and the holomorphic angle. Consider the normal vector fields

$$\begin{aligned} e_3 &= i\csc\alpha e_1 - \cot\alpha v \\ e_4 &= \cot\alpha e_1 + i\csc\alpha v \\ e_5 &= \csc\beta\xi - \cot\beta e_2 \end{aligned} \tag{3}$$

where $\beta \neq 0, \pi$ and $\alpha \neq 0, \pi$. We will call $(e_j)_{1\leq\ j\leq 5}$ an *adapted frame*.

Using (2) and (3), we get

$$\begin{aligned} v = \sin\beta e_2 - \cos\beta e_5, \quad iv &= \sin\alpha e_4 - \cos\alpha e_1 \\ \xi &= \cos\beta e_2 + \sin\beta e_5 \end{aligned} \tag{4}$$

It follows from (3) and (4) that

$$\begin{aligned} ie_1 &= \cos\alpha\sin\beta e_2 + \sin\alpha e_3 - \cos\alpha\cos\beta e_5 \\ ie_2 &= -\cos\beta z - \cos\alpha\sin\beta e_1 + \sin\alpha\sin\beta e_4 \end{aligned} \tag{5}$$

Consider now the dual basis (θ^j) of (e_j). The connection forms (θ^j_k) are given by

$$De_j = \theta^k_j e_k,$$

and the second fundamental form with respect to this frame are given by

$$II^j \ = \ \theta^j_1\theta^1 + \theta^j_2\theta^2; \quad j = 3, ..., 5.$$

Using (5) and differentiating v and ξ on the surface S, we get

$$\begin{aligned} D\xi &= -\cos\alpha\sin\beta\theta^2 e_1 + \cos\alpha\sin\beta\theta^1 e_2 + \sin\alpha\theta^1 e_3 + \sin\alpha\sin\beta\theta^2 e_4 \\ &\quad -\cos\alpha\cos\beta\theta^1 e_5, \qquad (6) \\ Dv &= (\sin\beta\theta^1_2 - \cos\beta\theta^1_5)e_1 + \cos\beta(d\beta - \theta^2_5)e_2 + (\sin\beta\theta^3_2 - \cos\beta\theta^3_5)e_3 \\ &\quad +(\sin\beta\theta^2_4 - \cos\beta\theta^4_5)e_4 + \sin\beta(d\beta + \theta^5_2)e_5. \end{aligned}$$

Differentiating e_3, e_4 and e_5, we have

$$\begin{aligned}
\theta_3^1 &= -\theta_1^3 \\
\theta_3^2 &= \sin\beta(d\alpha + \theta_4^1) - \cos\beta\sin\alpha\theta^1 \\
\theta_3^4 &= \csc\beta\theta_1^2 - \cot\alpha(\theta_1^3 + \csc\beta\theta_2^4) \\
\theta_3^5 &= \cot\beta\theta_2^3 - \csc\beta\sin\alpha\theta^1 \\
\theta_4^1 &= -d\alpha - \csc\beta\theta_2^3 + \sin\alpha\cot\beta\theta^1 \\
\theta_4^2 &= -\theta_2^4 \\
\theta_4^3 &= \csc\beta\theta_2^1 + \cot\alpha(\theta_1^3 + \csc\beta\theta_2^4) \\
\theta_4^5 &= \cot\beta\theta_2^4 - \sin\alpha\theta^2 \\
\theta_5^1 &= -\cos\alpha\theta^2 - \cot\beta\theta_2^1 \\
\theta_5^2 &= d\beta + \cos\alpha\theta^1 \\
\theta_5^3 &= -\cot\beta\theta_2^3 + \csc\beta\sin\alpha\theta^1 \\
\theta_5^4 &= -\cot\beta\theta_2^4 + \sin\alpha\theta^2
\end{aligned} \tag{7}$$

The conditions of minimality and of symmetry are equivalent to the following equations:

$$\theta_1^\lambda \wedge \theta^1 + \theta_2^\lambda \wedge \theta^2 = 0 = \theta_1^\lambda \wedge \theta^2 - \theta_2^\lambda \wedge \theta^1. \tag{8}$$

On the surface S, we consider

$$\theta_1^3 = a\theta^1 + b\theta^2$$

It follows from (8) that

$$\begin{aligned}
\theta_1^3 &= a\theta^1 + b\theta^2 \\
\theta_2^3 &= b\theta^1 - a\theta^2 \\
\theta_1^4 &= d\alpha + (b\csc\beta - \sin\alpha\cot\beta)\theta^1 - a\csc\beta\theta^2 \\
\theta_2^4 &= d\alpha \circ J - a\csc\beta\theta^1 - (b\csc\beta - \sin\alpha\cot\beta)\theta^2 \\
\theta_1^5 &= d\beta \circ J - \cos\alpha\theta^2 \\
\theta_2^5 &= -d\beta - \cos\alpha\theta^1
\end{aligned} \tag{9}$$

where J is the complex structure of S is given by $Je_1 = e_2$ and $Je_2 = -e_1$. Moreover, the normal connection forms are given by:

$$\begin{aligned}
\theta_3^4 &= -\sec\beta d\beta \circ J - \cot\alpha\csc\beta d\alpha \circ J + a\cot\alpha\cot^2\beta\theta^1 \\
&\quad +(b\cot\alpha\cot^2\beta - \cos\alpha\cot\beta\csc\beta + 2\sec\beta\cos\alpha)\theta^2 \\
\theta_3^5 &= (b\cot\beta - \csc\beta\sin\alpha)\theta^1 - a\cot\beta\theta^2 \\
\theta_4^5 &= \cot\beta(d\alpha \circ J) - a\cot\beta\csc\beta\theta^1 + (-b\csc\beta\cot\beta + \sin\alpha(\cot^2\beta - 1))\theta^2,
\end{aligned} \tag{10}$$

while the Gauss equation is equivalent to the equation:

$$d\theta_2^1 + \theta_k^1 \wedge \theta_2^k \;=\; \theta^1 \wedge \theta^2. \tag{11}$$

Therefore, using equations (9) and (11), we have

$$\begin{aligned}
K \;=\; & 1 - |\nabla\beta|^2 - 2\cos\alpha\beta_1 - \cos^2\alpha - (1+\csc^2\beta)(a^2+b^2) \\
& +2b\sin\alpha\csc\beta\cot\beta + 2\sin\alpha\cot\beta\alpha_1 - |\nabla\alpha|^2 \\
& +2a\csc\beta\alpha_2 - 2b\csc\beta\alpha_1 - \sin^2\alpha\cot^2\beta \\
& = 1 - (1+csc^2\beta)(a^2+b^2) - 2b\csc\beta(\alpha_1 - \sin\alpha\cot\beta) + 2a\csc\beta\alpha_2 \\
& -|\nabla\beta + \cos\alpha e_1|^2 - |\nabla\alpha - \sin\alpha\cot\beta e_1|^2
\end{aligned} \tag{12}$$

Using (7) and the complex structure of S, we get

$$\theta_2^1 \;=\; \tan\beta(d\beta\circ J - 2\cos\alpha\theta^2) \tag{13}$$

Differentiating (13), we conclude that

$$\begin{aligned}
d\theta_2^1 \;=\; & (-(1+\tan^2\beta)|\nabla\beta|^2 - \tan\beta\Delta\beta - 2\cos\alpha(1+2\tan^2\beta)\beta_1 \\
& +2\tan\beta\sin\alpha\alpha_1 - 4\tan^2\beta\cos^2\alpha)\theta^1\wedge\theta^2
\end{aligned}$$

where $\Delta = tr\nabla^2$ is the Laplacian of S. The Gaussian curvature is therefore given by:

$$\begin{aligned}
K \;=\; & -(1+\tan^2\beta)|\nabla\beta|^2 - \tan\beta\Delta\beta - 2\cos\alpha(1+2\tan^2\beta)\beta_1 \\
& +2\tan\beta\sin\alpha\alpha_1 - 4\tan^2\beta\cos^2\alpha.
\end{aligned} \tag{14}$$

From (12) and (14), we obtain the following formula for the Laplacian of S:

$$\begin{aligned}
\tan\beta\Delta\beta \;=\; & (1+\csc^2\beta)(a^2+b^2) + 2b\csc\beta(\alpha_1 - \sin\alpha\cot\beta) - 2a\csc\beta\alpha_2 \\
& -\tan^2\beta(|\nabla\beta + 2\cos\alpha e_1|^2 - |\cot\beta\nabla\alpha + \sin\alpha(1-\cot^2\beta)e_1|^2) \\
& +\sin^2\alpha(1-\tan^2\beta)
\end{aligned} \tag{15}$$

4 Gauss-Codazzi-Ricci equations for totally real minimal surfaces in S^5 with null principal curvatures

Definition 4.1. *We define a totally real minimal surface as a minimal surface in S^5 with Holomorphic angle $\alpha = \pi/2$.*

Using the connection form (9),(10) and (13) with $\alpha = \pi/2$ and a, b nulls, we have

$$\begin{array}{ccl} \theta_1^3 & = 0 = & \theta_2^3 \\ \theta_1^4 & = & -\cot\beta\theta^1 \\ \theta_2^4 & = & \cot\beta\theta^2 \\ \theta_1^5 & = & \beta_2\theta^1 - \beta_1\theta^2 \\ \theta_2^5 & = & -\beta_1\theta^1 - \beta_2\theta^2 \\ \theta_1^5 & = & \beta_2\theta^1 - \beta_1\theta^2 \\ \theta_3^4 & = & -\sec\beta(\beta_2\theta^1 - \beta_1\theta^2) \\ \theta_3^5 & = & -\csc\beta\theta^1 \\ \theta_4^5 & = & \beta_2\theta^1 - \beta_1\theta^2 \\ \theta_2^1 & = & \tan\beta(\beta_2\theta^1 - \beta_1\theta^2) \end{array} \tag{16}$$

Now Codazzi-Ricci equations:

$$\begin{array}{rcl} d\theta_1^3 + \theta_2^3 \wedge \theta_1^2 + \theta_4^3 \wedge \theta_1^4 + \theta_5^3 \wedge \theta_1^5 & = & 0 \\ d\theta_2^4 + \theta_1^4 \wedge \theta_2^1 + \theta_3^4 \wedge \theta_2^3 + \theta_5^4 \wedge \theta_2^5 & = & 0 \\ d\theta_4^5 + \theta_1^5 \wedge \theta_4^1 + \theta_2^5 \wedge \theta_4^2 + \theta_3^5 \wedge \theta_4^3 & = & 0 \end{array}$$

simplify to:

$$2\csc\beta\beta_1 = 0 \tag{17}$$

Therefore:

$$\beta_1 = 0 \tag{18}$$

The following Codazzi-Ricci equations are always verified:

$$\begin{array}{rcl} d\theta_2^3 + \theta_1^3 \wedge \theta_2^1 + \theta_4^3 \wedge \theta_2^4 + \theta_5^3 \wedge \theta_2^5 & = & 0 \\ d\theta_1^4 + \theta_2^4 \wedge \theta_1^2 + \theta_3^4 \wedge \theta_1^3 + \theta_5^4 \wedge \theta_1^5 & = & 0 \\ d\theta_3^5 + \theta_1^5 \wedge \theta_3^1 + \theta_2^5 \wedge \theta_3^2 + \theta_4^5 \wedge \theta_3^4 & = & 0 \\ d\theta_2^5 + \theta_1^5 \wedge \theta_2^1 + \theta_3^5 \wedge \theta_2^3 + \theta_4^5 \wedge \theta_2^4 & = & 0 \end{array}$$

Remark 4.2. *The Contact angle(β) is determined just only in direction of e_1, and it is constant function in this direction.*

Gauss-Codazzi-Ricci equations:

$$\begin{array}{rcl} d\theta_1^2 + \theta_3^2 \wedge \theta_1^3 + \theta_4^2 \wedge \theta_1^4 + \theta_5^2 \wedge \theta_1^5 & = & \theta^2 \wedge \theta^1 \\ d\theta_1^5 + \theta_2^5 \wedge \theta_1^2 + \theta_3^5 \wedge \theta_1^3 + \theta_4^5 \wedge \theta_1^4 & = & 0 \\ d\theta_3^4 + \theta_1^4 \wedge \theta_3^1 + \theta_2^4 \wedge \theta_3^2 + \theta_5^4 \wedge \theta_3^5 & = & 0 \end{array}$$

give the following Laplacian equation of the Contact angle (β):

$$\tan\beta\Delta(\beta) = -\tan^2\beta|\nabla\beta|^2 + \cot^2\beta - 1 \tag{19}$$

5 Proof of the Theorem 1.1

In this section, we will give a prove of the theorem, using Gauss-Codazzi-Ricci equations for a compact minimal surface in S^5 with constant holomorphic angle ($\alpha = \pi/2$) and null principal curvatures a, b.

Now suppose that ($\pi/4 \leq \beta < \pi/2$) at the equation (19), then $\Delta\beta < 0$ and using the Hopf's Lemma, we get the Theorem (1.1).

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