Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 24, 1193 - 1199

Domination in Jahangir Graph $J_{2,m}$

D. A. Mojdeh and A. N. Ghameshlou

Department of Mathematics
University of Mazandaran
P.O. Box 47416-1467, Babolsar, Iran
dmojdeh@umz.ac.ir
a.ghameshlou@umz.ac.ir

Abstract

Given graph $G = (V, E)$, a dominating set S is a subset of vertex set V such that any vertex not in S is adjacent to at least one vertex in S. The domination number of a graph G is the minimum size of the dominating sets of G. In this paper we study some results on domination number, connected, independent, total and restrained domination number denoted by $\gamma(G)$, $\gamma_c(G)$,$\gamma_i(G)$, $\gamma_t(G)$ and $\gamma_r(G)$ respectively in Jahangir graphs $J_{2,m}$.

Keywords: Jahangir graphs; domination number; independent; total; restrained

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A dominating set is a set S of vertices such that every vertex outside S is dominated by some vertex of S. The domination number of G, denoted by $\gamma(G)$, is the minimum size of dominating set of G. A dominating set S is called a connected dominating set if $G[S]$, induced subgraph by S, is connected. The connected domination number of G is the minimum size of connected dominating set of G and is denoted by $\gamma_c(G)$.

A dominating set S is called an independent dominating set if S is an independent set. The independent domination number of G denoted by $\gamma_i(G)$ is the minimum size of an independent dominating set of G. A dominating set S is a total dominating set of G if $G[S]$ has no isolated vertex, i.e. every vertex in S has at least an adjacent in S.

The total domination number of G, denoted by $\gamma_t(G)$ is the minimum size of a total dominating set of G. The restrained dominating set of G is a subset S of V that every $v \in V - S$ is adjacent to at least one other vertex in

$V - S$. Every graph has a restrained dominating set, since $S = V$ is such a set. Restrained domination number denoted by $\gamma_r(G)$ is the minimum size of the restrained dominating sets of G. Clearly $\gamma_r(G) \geq \gamma(G)$, (see [2,3,4,5]).

Definition 1 *Jahangir graphs $J_{n,m}$ for $m \geq 3$,is a graph on $nm + 1$ vertices i.e. , a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}, (see[1]).*

In this paper we study dominating number, connected, total, independent and restrained dominating number for Jahangir graphs $J_{2,m}$.

Example 2 *Figure* 1 *shows Jahangir graph $J_{2,8}$. The figure1, $J_{2,8}$, appears on Jahangir's tomb in his mausoleum. It lies in 5 kilometer north-west of Lahore, Pakistan across the River Ravi, (see [1]).*

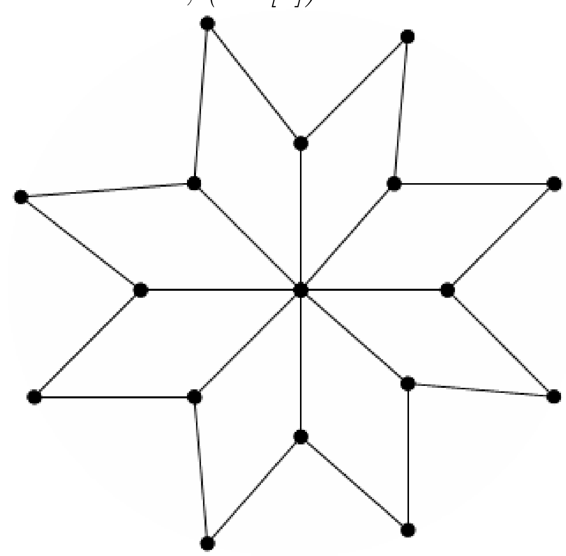

Figure 1: $J_{2,8}$

Example 3 $\gamma(J_{2,3}) = 2$.

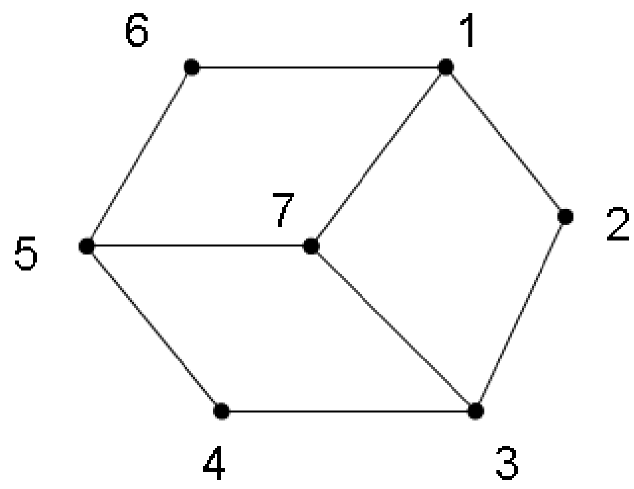

Figure 2: $J_{2,3}$

The following theorems are useful:

Theorem A(see[5]): If G has no isolated vertices, then $\gamma(G) \leq \frac{|V(G)|}{2}$.

Theorem B(see[5]): (a) If a graph G has no isolated vertices, $\gamma_t(G) \leq n - \Delta(G) + 1$. (b) If a graph G is connected and $\Delta(G) < n - 1$, then $\gamma_t(G) \leq n - \Delta(G)$.

2 Domination Number, Connected and Total Domination Number of $J_{2,m}$

In this section we study domination number, connected and total domination number of Jahangir graphs $J_{2,m}$.

Remark 4 *Let* v_{2m+1} *be the label of the center vertex and* $v_1, v_2, ..., v_{2m}$ *be the label of the vertices that incident clockwise on cycle* C_{2m} *so that* $deg(v_1) = 3$.

Theorem 5 *For* $m \geq 4$, $\gamma(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$.

Proof. Suppose that $m = 2k$, for some positive integer k. It is easy to verify that the set of vertices

$$S_0 = \{v_1, v_5, \ldots , v_{2m-3}, v_{2m+1}\}$$

is a dominating set for $J_{2,m}$. Therefore $\gamma(J_{2,m}) \leq |S_0| = \frac{m}{2} + 1$. Let $D \subset V(J_{2,m})$, $|D| \leq \frac{m}{2}$ and D be a dominating set of $J_{2,m}$.

We consider three following cases:

Case 1. Let $v_{2m+1} \in D$ and $\frac{m}{2}-1$ remained vertices of D be incident odd index vertices of cycle C_{2m}. According to the structure of labeling, it is clear that vertex v_{2m+1} dominates all odd index vertices, i.e. $v_1, v_3, \dots, v_{2m-1}$. Therefore $D - \{v_{2m+1}\}$ must dominate m even index vertices, i.e. $v_2, v_4, \dots, v_{2m}$. So, at most $m-2$ even index vertices are dominated by $\frac{m}{2}-1$ odd index vertices of cycle C_{2m}. Therefore, we have a contradiction since there is at least two even index vertices on $V(J_{2,m})$ that any vertices of D can not dominate them.

Case 2. Let $v_{2m+1} \in D$ and $\frac{m}{2}-1$ remained vertices of D be incident even index vertices of cycle C_{2m}. According to the structure of labeling it is clear that vertex v_{2m+1} dominates all odd index vertices. In this case D must dominate m even index vertices with $\frac{m}{2}-1$ even index vertices. It is impossible. Because there is not any even index vertices in $J_{2,m}$ that dominates other even index vertices. So, $\frac{m}{2}+1$ even index vertices are not dominated by D and it is a contradiction.

Case 3. Let $v_{2m+1} \notin D$ and $\frac{m}{2}$ of the incident vertices of cycle C_{2m} be in D. Without loss of generality, suppose that $v_1 \in D$. Then v_1 dominates itself and the vertices v_{2m+1}, v_{2m} and v_2, and $\frac{m}{2}-1$ remained vertices of D dominate at most $m-2$ vertices that are not dominated yet. So, we have a contradiction. Since at least $\frac{m}{2}$ of vertices in $V(J_{2,m})$ are not dominated by vertices in D. Hence, $\gamma(J_{2,m}) \geq \frac{m}{2}+1$. This implies that $\gamma(J_{2,m}) = \frac{m}{2}+1$.

Now suppose that for some positive integer k, $m = 2k+1$.

It is easy to verify that the set of vertices $S_1 = \{v_1, v_5, v_9, ..., v_{2m-1}, v_{2m+1}\}$ is a dominating set for $J_{2,m}$. Therefore $\gamma(J_{2,m}) \leq |S_1| = \lceil \frac{m}{2} \rceil + 1$. So, it is sufficient to show that $\gamma(J_{2,m}) \geq \lceil \frac{m}{2} \rceil + 1$. Let $D \subset V(J_{2,m})$, $|D| \leq \lceil \frac{m}{2} \rceil$ and D be a dominating set of $J_{2,m}$.

We consider three following cases:

Case 1. Let $v_{2m+1} \in D$ and $\frac{m-1}{2}$ odd index vertices of cycle C_{2m} be in D. According to structure of labeling it is clear that vertex v_{2m+1} dominates all odd index vertices. Therefore D must dominate m even index vertices with $\frac{m-1}{2}$ odd index vertices of cycle C_{2m}. So, at most $m-1$ even index vertices are dominated by $\frac{m-1}{2}$ odd index vertices of cycle C_{2m}. Hence, we have a contradiction since there is at least one even index vertex on $V(J_{2,m})$ that any vertices of D cannot dominate it.

Case 2. Let $v_{2m+1} \in D$ and $\frac{m-1}{2}$ even index vertices of cycle C_{2m} be in D. In this case D must dominate m even index vertices with $\frac{m-1}{2}$ even index vertices. It is impossible. Because there are not any even index vertices in $J_{2,m}$ that dominate other even index vertices. So, $\frac{m+1}{2}$ even index vertices are not dominated by D and it is a contradiction.

Case 3. Let $v_{2m+1} \notin D$ and $\lceil \frac{m}{2} \rceil$ incidence vertices of the cycle C_{2m} be in D. Without loss of generality, suppose that $v_1 \in D$. Then v_1 dominates itself and the vertices v_{2m+1}, v_{2m} and v_2, and $\frac{m-1}{2}$ remained vertices of D dominate at most $m-1$ vertices that are not dominated yet. So, we have a contradiction. Since at least $\frac{m-3}{2}$ of vertices in $V(J_{2,m})$ are not dominated by vertices in D. Therefore $\gamma(J_{2,m}) \geq \lceil \frac{m}{2} \rceil + 1$. This implies that $\gamma(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$. ■

Corollary 6 $\gamma_c(J_{2,m}) = \gamma(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$.

Proof. Obviuesly $\gamma_c(J_{2,m}) \geq \gamma(J_{2,m})$. In theorem 5, we showed that the sets S_0 and S_1 are optimal dominating sets of $J_{2,m}$. Induced subgraphs by the sets S_0 and S_1, $G[S_0]$ and $G[S_1]$ respectively, are a Star, i.e. $K_{1,\lceil \frac{m}{2} \rceil}$. Clearly $G[S_0]$ and $G[S_1]$ are connected and S_0 and S_1 are optimal connected dominating sets of $J_{2,m}$. Hence, $\gamma_c(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$. ■

By Theorem B we know that $\gamma_t(J_{2,m}) \leq m+1$. Now we optimized this upper bound and show that $\gamma_t(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$.

Corollary 7 $\gamma_t(J_{2,m}) = \gamma(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$.

Proof. Clearly, $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$. It is trivial by Theorem 5 and Corollary 6. ■

3 Independent and Restrained Domination Number of $J_{2,m}$

In this section we study $\gamma_i(J_{2,m})$ and $\gamma_r(J_{2,m})$.

Theorem 8 *For $m \geq 3$, $\gamma_i(J_{2,m}) = \lceil \frac{2m}{3} \rceil$.*

Proof. Suppose that $m \equiv 0$ $(mod\ 3)$. It is easy to verify that the set of vertices
$S_0 = \{v_1, v_4, v_7, \ldots, v_{2m-5}, v_{2m-2}\}$ is an independent dominating set of $J_{2,m}$. Let $2m \equiv 1$ $(mod\ 3)$. It is easy to see that $S_1 = \{v_1, v_4, v_7, \ldots, v_{2m-3}, v_{2m-1}\}$ is an independent dominating set of $J_{2,m}$. Finally if $2m \equiv 2$ $(mod\ 3)$, the set of vertices $S_2 = \{v_1, v_4, v_7, \ldots, v_{2m-4}, v_{2m-1}\}$ is an independent dominating set of $J_{2,m}$. Hence, $\gamma_i(J_{2,m}) \leq |S_0| = |S_1| = |S_2| = \lceil \frac{2m}{3} \rceil$. Let $D \subset V(J_{2,m})$, $|D| \leq \lceil \frac{2m}{3} \rceil - 1$ and D be an independent dominating set of $J_{2,m}$. We consider two following cases:

Case 1. Let $v_{2m+1} \in D$.

According to the structure of labeling, it is clear that vertex v_{2m+1} dominates itself and all odd index vertices. Since the set D is an independent dominating set, any odd index vertices can not be in D. So, D must dominate m even index vertices with $\lceil \frac{2m}{3} \rceil - 2$ even index vertices. It is impossible. Because there are not any even index vertices in $J_{2,m}$ that dominate other even index vertices. Therefore we have a contradiction and $\gamma_i(J_{2,m}) \geq \lceil \frac{2m}{3} \rceil$.

Case 2. Let $v_{2m+1} \notin D$.

Without loss of generality, suppose that $v_1 \in D$. The vertex v_1 dominates itself and the vertices v_{2m+1}, v_{2m} and v_2, and $\lceil \frac{2m}{3} \rceil - 2$ remained vertices of D can dominate independently at most $2\lceil \frac{2m}{3} \rceil - 4$ vertices that are not dominated yet. So, there is at least one vertex that is not dominated by the independent dominating set D. It is a contradiction and $\gamma_i(J_{2,m}) \geq \lceil \frac{2m}{3} \rceil$.

These cases imply that $\gamma_i(J_{2,m}) = \lceil \frac{2m}{3} \rceil$. ∎

Theorem 9 *For $m \geq 4$ and some positive integer k,*
$$\gamma_r(J_{2,m}) = \begin{cases} \lceil \frac{m}{2} \rceil + 1 & \text{if} \quad m = 2k \\ \lceil \frac{m}{2} \rceil + 2 & \text{otherwise} \end{cases}.$$

Proof. Clearly, $\gamma_r(J_{2,m}) \geq \gamma(J_{2,m})$.

We consider two following cases:

Case 1. Let $m = 2k$.

In Theorem 5 we showed that if $m = 2k$, the set $S_0 = \{v_1, v_5, \dots, v_{2m-3}, v_{2m+1}\}$ is a dominating set for $J_{2,m}$. Obviously $V - S_0 = \{v_j | j = 2q\} \bigcup \{v_3, v_7, \dots v_{2m-5}, v_{2m-1}\}$. It is easy to see that every vertex in $V - S_0$ has at least an adjacent vertex in $V - S_0$. So, the set S_0 is a restrained dominating set of $J_{2,m}$. Therefore $\gamma_r(J_{2,m}) \leq \gamma(J_{2,m})$. This implies that $\gamma_r(J_{2,m}) = \gamma(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$.

Case 2. Let $m = 2k + 1$.

In Theorem 5 we showed that if $m = 2k + 1$, the set of vertices $S_1 = \{v_1, v_5, v_9, ..., v_{2m-1}, v_{2m+1}\}$ is a dominating set for $J_{2,m}$. Obviously $V - S_1 = \{v_j | j = 2q\} \bigcup \{v_3, v_7, \dots v_{2m-7}, v_{2m-3}\}$. The set S_1 dominates all vertices in the $V - S_1$ of $J_{2,m}$ restrainedly except the center vertex v_{2m}. Since the center vertex v_{2m} in the set $V - S_1$ has no adjacent vertex. Hence, $S_2 = S_1 \bigcup \{v_{2m}\}$ is a restrained dominating set of $J_{2,m}$ and $|S_2| = \lceil \frac{m}{2} \rceil + 2$ and proof is completed. ∎

References

[1] K. Ali, E. T. Baskoro and I. Tomescu, *On the Ramzey number of Paths and Jahangir graph $J_{3,m}$*. 3rd International Conference on 21st Century Mathematics 2007, GC University Lahoor Pakistan, March 2007.

[2] G. S. Domke, J. H. Hattingh, S. T. Hedetniemi, R. C. Laskar, and L. R. Markus, *Restrained domination in graphs*, Discrete Math. **203**(1999), 61-69.

[3] T. W. Haynes, S. T. Hedetniem, P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New york, 1998.

[4] S. T. Hedetniemi, R. Laskar, *Domination in graphs*, Graph Theory and Combinatorics: Proceedings of the Cambridge Combinatorical Conference, Academic Press, London, 1984.

[5] B. Liu, *Some inequalities about connected domination number*, Disccrete Math **159** (1996) 241-245.

[6] D. A. Mojdeh, S. A. Hassanpoor, H. Abdollahzadeh Ahangar and A. Ahmadi Haji, *On Domination in Hamiltonian cubic graph*, Far East Journal of Mathemetical Sciences, vol **24** (2007), 187-200.

Received: March 1, 2007