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Fixed Point Results for Multivalued Maps

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Abstract

Using the concept of w-distance, we prove some results on the existence of fixed points and common fixed points for multivalued Kannan maps. Consequently, we improve and generalize the corresponding fixed point results due to Latif and Beg, Suzuki, Kannan and many others.

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1 Introduction

The well known Banach contraction principle, which asserts that "each single-valued contraction self map on a complete metric space has a unique fixed point" has been generalized in many different directions. Nadler [8] has used the concept of Hausdorff metric and obtained a multivalued version of the Banach contraction principle. Among others Husain and Latif [2], Feng and

Liu [1] have generalized Nadler's fixed point result without using the Hausdorff metric. On the other hand, Kannan [4] has proved an interesting fixed point result for single-valued maps in the setting of metric spaces which is not an extension of the Banach contraction principle. While, Latif and Beg [6] have obtained a multivalued version of the Kannan's fixed point result.

In [3], Kada et al. have introduced a notion of w-distance on a metric space and improved several results replacing the involved metric by a generalized distance. Using this generalized distance, Suzuki and Takahashi [11] have introduced notions of single-valued and multivalued weakly contractive maps and proved fixed point results for such maps. Consequently, they generalized the Banach Contraction principle and Nadler's fixed point result. While, Susuki [12] generalized Kannan's fixed point result under w-distance. Recent fixed point results concerning w-distance can be found in [5, 10, 12, 13, 15]

In this paper, using the concept of w-distance, we prove fixed point and common fixed point results for multivalued maps in the setting of metric spaces. Our results improve and generalize the corresponding results due to Latif and Beg [6], Suzuki [12], Kannan [4] and many others.

2 Preliminaries

Throughout this paper, X is a metric space with metric d. Let 2^X denote a collection of all nonempty subsets of X, $Cl(X)$ a collection of all nonempty closed subsets of X. Consider a single-valued map $f : X \to X$ and a multivalued map $T : X \to 2^X$. (a) An element $x \in X$ is called a *fixed point* of f if $x = f(x)$, and a fixed point of T if $x \in T(x)$. (b) f is called *Banach contraction* if for a fixed constant $h \in [0, 1)$ and for each $x, y \in X$, $d(f(x), f(y)) \leq h\ d(x, y)$. (c) f is called *Kannan contraction* if for a fixed constant $r \in [0, 1/2)$ and for each $x, y \in X$, $d(f(x), f(y)) \leq r\ \{d(x, f(x)) + d(y, f(y))\}$. Clearly, Kannan contraction (which may not be continuous) is not a generalization of the Banach contraction. Kannan [4] has proved that each Kannan contraction self map on a complete metrics has a unique fixed point.

A map $\varphi : X \to \mathbb{R}$ is called *lower semi-continuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $\varphi(x) \leq \lim inf_{n\to\infty} \varphi(x_n)$.

Recently, Kada et al. [3] introduced a concept of w-distance as follows:

A function $\omega : X \times X \to [0, \infty)$ is called *w-distance* on X if it satisfies the following for any $x,\ y,\ z \in X$:

(w_1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$;

(w_2) a map $\omega(x, .) : X \to [0, \infty)$ is lower semicontinuous;

(w_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

The metric d is a w-distance on X. Many other examples of w-distance are given in [3, 11, 13, 14]. Note that, in general for $x, y \in X$, $\omega(x, y) \neq \omega(y, x)$.

We say a single-valued map $f : X \to X$ is *w-Kannan* [12] if there exist a w-distance ω on X and $r \in [0, 1/2)$ such that for each $x, y \in X$, $\omega(f(x), f(y)) \leq r\{\omega(x, f(x)) + \omega(y, f(y))\}$. A multivalued map $T : X \to 2^X$ is K_w- map if there exists a nonnegative number $r \in [0, \frac{1}{2})$ and a w-distance function ω such that for any $x \in M$, $u \in T(x)$ there exists $v \in T(y)$ for all $y \in M$ such that

$$\omega(u, v) \leq r\{\omega(x, u) + \omega(y, v)\}.$$

In particular, if we take $\omega = d$, then w-Kannan map is Kannan contraction and K_w-map is K-map [6, 7].

The following lemma concerning w-distance is fundamental.

Lemma 2.1 [3] *Let X be a metric space with metric d and let ω be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$ Then, the following hold:*

`(a)` *If $\omega(x_n, y) \leq \alpha_n$ and $\omega(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$; in particular, if $\omega(x, y) = 0$ and $\omega(x, z) = 0$, then $y = z$;*

`(b)` *if $\omega(x_n, y_n) \leq \alpha_n$ and $\omega(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*

`(c)` *if $\omega(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

`(d)` *if $\omega(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

3 The Results

In this section, we consider X complete and M a nonempty closed subset of X.

Theorem 3.1 *Let $T : M \to Cl(M)$ be a multivalued K_w-map such that*

$$\inf\{\omega(x,u) + \omega(x,\ T(x)) : x \in X\} > 0,$$

for every $u \in X$ with $u \notin T(u)$. Then T has a fixed point.

Proof. Let u_o be an arbitrary element of M and let $u_1 \in T(u_o)$ be fixed. Since T is K_ω-map, there exists $u_2 \in T(u_1)$ such that

$$\omega(u_1, u_2) \leq r\,\omega(u_o, u_1) + r\,\omega(u_1, u_2),$$

where $r \in [0, \frac{1}{2})$ and consequently

$$\omega(u_1, u_2) \leq \frac{r}{1-r}\,\omega(u_o, u_1).$$

Thus, we get a sequence $\{u_n\}$ in M such that for every $n \in \mathbb{N}$, $u_{n+1} \in T(u_n)$ and

$$\omega(u_n, u_{n+1}) \leq [\frac{r}{1-r}]\,\omega(u_{n-1}, u_n),$$

for some fixed r, $0 < r < \frac{1}{2}$. Note that for any $n \in \mathbb{N}$, we have

$$\omega(u_n, u_{n+1}) \leq [\frac{r}{1-r}]^n\,\omega(u_o, u_1).$$

Put $\lambda = \frac{r}{1-r}$. Then $0 < \lambda < 1$. For m and n positive integers $m > n$, we have

$$\begin{aligned}\omega(u_n, u_m) &\leq \omega(u_n, u_{n+1}) + \omega(u_{n+1}, u_{n+2}) + ... + \omega(u_{m-1}, u_m),\\ &\leq \lambda^n\,\omega(u_o, u_1) + \lambda^{n+1}\omega(u_o, u_1) + ... + \lambda^{m-1}\omega(u_o, u_1),\\ &\leq \frac{\lambda^n}{1-\lambda}\omega(u_o, u_1),\end{aligned}$$

which implies that $\omega(u_n, u_m) \to 0$ as $n \to \infty$ and by Lemma 2.1, $\{u_n\}$ is a Cauchy sequence. From the completeness of X, we get that $\{u_n\}$ converges to some $v_o \in X$. M being closed we have $v_o \in M$. Let $n \in \mathbb{N}$ be fixed. Since $\{u_m\}$ converges to some v_o and $\omega(u_n, .)$ is lower semicontinuous, we have

$$\omega(u_n, v_o) \leq \liminf_{m\to\infty} \omega(u_n, u_m) \leq \frac{\lambda^n}{1-\lambda}\omega(u_o, u_1),$$

So, as $n \to \infty$, we have $\omega(u_n, v_o) \to 0$. Assume that $v_o \notin T(v_o)$. Then, by hypothesis, we have

$$\begin{aligned}0 &< \inf\{\omega(u, v_o) + \omega(u, T(u)) : u \in X\}\\ &\leq \inf\{\omega(u_n, v_o) + \omega(u_n, T(u_n)) : n \in \mathbb{N}\}\\ &\leq \inf\{\omega(u_n, v_o) + \omega(u_n, u_{n+1}) : n \in \mathbb{N}\}\\ &\leq \inf\{\frac{\lambda^n}{1-\lambda}\omega(u_o, u_1) + \lambda^n\omega(u_o, u_1) : n \in \mathbb{N}\} = 0,\end{aligned}$$

which is impossible and hence $v_o \in T(v_o)$.

Theorem 3.2 *Each K_w-map $T : M \to Cl(M)$ has a fixed point, provided that for any iterative sequence $\{u_n\}$ in M with $u_n \to v_o \in M$, the sequence of real numbers $\{\omega(v_o, u_n)\}$ converges to zero.*

Proof. Following the proof of Theorem 3.1 , there exists a convergent iterative sequence $\{u_n\}$ such that $u_n \to v_o \in M$ with

$$\omega(u_n, v_o) \leq \liminf_{m\to\infty} \omega(u_n, u_m) \leq \frac{\lambda^n}{1-\lambda}\omega(u_o, u_1),$$

and

$$\omega(u_n, u_{n+1}) \leq \lambda^n \omega(u_o, u_1).$$

where $\lambda = \frac{r}{1-r} < 1$. Note that $\omega(u_n, v_o) \to 0$ as $n \to \infty$. Further, since $u_n \in T(u_{n-1})$ and T is a K_ω-map, there is $v_n \in T(v_o)$ such that

$$\begin{aligned} \omega(u_n, v_n) &\leq r\,[\omega(u_{n-1}, u_n) + \omega(v_o, v_n)] \\ &\leq r\,\omega(u_{n-1}, u_n) + r\,\omega(v_o, u_n) + r\,\omega(u_n, v_n) \\ &\leq \frac{r}{1-r}\omega(u_{n-1}, u_n) + \frac{r}{1-r}\omega(v_o, u_n). \end{aligned}$$

and thus $\omega(u_n, v_n) \to 0$, as $n \to \infty$ Thus, by Lemma 2.1, we get that $v_n \to v_o$ and since $v_n \in T(v_o)$, which is closed, so $v_o \in T(v_o)$.

Now, we prove some results on the existence of common fixed points.

Theorem 3.3 *Let $\{T_n\}$ be a sequence of multivalued maps of M into $Cl(M)$. Suppose that there exists a constant $0 \leq r < \frac{1}{2}$ such that for any two maps $T_i, T_j \in \{T_n\}$ and for any $x \in M$, $u \in T_i(x)$, there exists $v \in T_j(y)$ for all $y \in M$ with*

$$\omega(u, v) \leq r\,\{\omega(x, u) + \omega(y, v)\},$$

and for each $n \geq 1$

$$\inf\{\omega(x, u) + \omega(x, T_n(x)) : x \in X\} > 0,$$

for any $u \notin T_n(u)$. Then, $\{T_n\}$ has a common fixed point.

Proof. Let u_o be an arbitrary element of M and let $u_1 \in T_1(u_o)$. Then, there is an $u_2 \in T_2\ (u_1)$ such that

$$\omega(u_1, u_2) \leq \frac{r}{1-r}\omega(u_o, u_1).$$

So, there exists a sequence $\{u_n\}$ such that $u_{n+1} \in T_{n+1}(u_{\ n})$ and for all $n \geq 1$,

$$\omega(u_n, u_{n+1}) \leq [\frac{r}{1-r}]^n \omega(u_o, u_1).$$

Put $\lambda = \frac{r}{1-r}$. Note that $0 < \lambda < 1$ and

$$\omega(u_n, u_{n+1}) \leq \lambda^n \omega(u_o, u_1),$$

for all $n \geq 1$. Then, as $n \to \infty$, we get that $\{u_n\}$ is a Cauchy sequence in X. Let $p = \lim_{n\to\infty} u_n$ in M. Now we show that $p \in \cap_{n\geq 1} T_n(p)$. Let T_m be an arbitrary member of $\{T_n\}$. Since $u_n \in T_n(u_{n-1})$, by hypothesis there is $s_n \in T_m(p)$ such that

$$\omega(u_n, s_n) \leq r\,\{\omega(u_{n-1}, u_n) + \omega(p, s_n)\}.$$

We proceed as in the proof of Theorem 3.1 and get

$$\omega(u_n, p) \leq \liminf_{m\to\infty} \omega(u_n, u_m) \leq \frac{\lambda^n}{1-\lambda}\omega(u_o, u_1),$$

which is converges to 0 as $n \to \infty$. Now, assume that $p \notin T_m(p)$. Then, by hypothesis, and for $n > m$ and $m \geq 1$ we have

$$\begin{aligned} 0 &< \inf\{\omega(u, p) + \omega(u, T_m(u)) : u \in X\} \\ &\leq \inf\{\omega(u_{m-1}, p) + \omega(u_{m-1}, T_m(u_{m-1})) : m \in \mathbb{N}\} \\ &\leq \inf\{\omega(u_{m-1}, p) + \omega(u_{m-1}, u_m) : m \in \mathbb{N}\} \\ &\leq \inf\{\frac{\lambda^{m-1}}{1-\lambda}\omega(u_o, u_1) + \lambda^{m-1}\omega(u_o, u_1) : m \in \mathbb{N}\} = 0. \end{aligned}$$

which is impossible and hence $p \in T_m(p)$. But T_m is an arbitrary, hence p is a common fixed point.

Theorem 3.4 *Let $\{T_n\}$ be a sequence of multivalued maps of M into $Cl(M)$. Suppose that there exists a constant r with $0 \leq r < \frac{1}{2}$ and such that For any two maps T_i, T_j and for any $x \in M$, $u \in T_i(x)$ there exists $v \in T_j(y)$ for all $y \in M$ with*

$$\omega(u, v) \leq r\,\{\omega(x, u) + \omega(y, v)\}.$$

Then $\{T_n\}$ has a common fixed point, provided that, for any iterative sequence $\{u_n\}$ in M with $u_n \to v_o \in M$, the sequence of real numbers $\{\omega(v_o, u_n)\}$ converges to zero.

Proof. By a similar method as in the proof of Theorem 3.3, the result follows.

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