Banach-Steinhaus Type Theorems in Locally Convex Spaces for LSC Convex Processes

S. Lahrech, J. Hlal, A. Ouahab
Dept. of Mathematics, Faculty of Science, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

A. Jaddar
National School of Management, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

A. Mbarki
National School of Applied Sciences, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

Abstract
Banach-Steinhaus type results are established for LSC convex processes between locally convex spaces.

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1 Introduction
Let \((X, \lambda)\) and \((Y, \mu)\) be two locally convex spaces. Assume that the locally convex topology \(\mu\) is generated by the family \(\{q_\beta\}_{\beta \in I}\) of semi norms on \(Y\). Let \(\beta(X, \lambda)\) denote the family of bounded sets in \((X, \lambda)\) and let \(\sigma \subset \beta(X, \lambda)\) such that \(\bigcup_{C \in \sigma} C = X\). An operator \(T: (X, \lambda) \to (Y, \mu)\) is said to be sequentially continuous if for every sequence \((x_n)\) of \(X\) and every \(x \in X\) such that \(x_n \xrightarrow{\lambda} x\) one has \(T x_n \xrightarrow{\mu} Tx\).
It has been shown in [1] that if \( T_n : (X, \lambda) \to (Y, \mu) \) is a sequence of sequentially continuous linear operators satisfying \( \forall C \in \sigma \lim T_n y = Ty \) uniformly in \( y \in C \) with respect to the topology \( \mu \), then the limit operator \( T \) sends \( \lambda \)-bounded sets into \( \mu \) bounded sets. Our main objective in this paper is to generalize some results established by S. Lahrech in [1] to LSC convex processes.

Recall that a convex process \( \Phi : X \to Y \) satisfying the condition: \( \forall x_0 \in X, \Phi(x_0) \subset (\lambda, \mu) - \limsup_{x \to x_0} \Phi(x) \equiv \{ y \in Y : \forall x_n \xrightarrow{\lambda} x_0, \exists (x_{n_k}) \text{ subsequence of } (x_n) \exists y_k \in \Phi(x_{n_k}) \text{ such that } y_k \xrightarrow{\mu} y \} \)

is called a \((\lambda, \mu)\)-LSC convex process.

Assume that \( C(X_\lambda) \subset \sigma \), where \( C(X_\lambda) \) denote the class of sequentially relatively compacts sets in \((X, \lambda)\). Denote by \( LSC_0((X, \lambda), R) \) the class of LSC convex processes at 0 acting from \((X, \lambda)\) into \( R \).

2 Banach-Steinhaus theorem in locally convex spaces for LSC convex processes

Assume that all the hypotheses of the above paragraph are satisfied. Let \( \Phi : X \to Y \) be a multifunction. The domain of \( \Phi \) is the set

\[
D(\Phi) = \{ x \in X : \Phi(x) \neq \emptyset \}.
\]

We say \( \Phi \) has nonempty images if its domain is \( X \). For any subset \( C \) of \( X \) we write \( \Phi(C) \) for the image \( \bigcup_{x \in C} \Phi(x) \) and the range of \( \Phi \) is the set \( R(\Phi) = \Phi(X) \).

We say \( \Phi \) is surjective if its range is \( Y \). The graph of \( \Phi \) is the set

\[
G(\Phi) = \{ (x, y) \in X \times Y : y \in \Phi(x) \}.
\]

A multifunction is convex, or closed if its graph is likewise.

A process is a multifunction whose graph is a cone. For example, we can interpret linear closed operators as closed convex processes.

A convex process \( \Phi : X \to Y \) is said to be \((\lambda, \mu)\) -bounded if it maps every bounded set of \((X, \lambda)\) into bounded set in \((Y, \mu)\).

Denote by \( \mathcal{M}(X, Y) \) the class of multifunctions acting from \( X \) into \( Y \).

Let also \( LSC_0((X, \lambda), (Y, \mu)) \) denote the class of LSC convex processes at 0 from \((X, \lambda)\) into \((Y, \mu)\).

For \( A \subset Y, B \subset Y \) with \( A \neq \emptyset \) and \( B \neq \emptyset \), we define the direct Hausdorff distance \( \rho_\beta \) with respect to the semi norm \( q_\beta \) by: \( \rho_\beta(A, B) = \sup_{y \in B} d^\beta_A(y) \), where \( d^\beta_A \) is the distance function to the set \( A \) with respect to the semi norm \( q_\beta \). If
A = ∅ or B = ∅, then we set ρß (A, B) = 0.
For Φ1, Φ2 ∈ 𝒜(X, Y) and for C ∈ σ, β ∈ I, set

\[ \nu_{C,\beta}(\Phi_1, \Phi_2) = \sup_{x \in C} \rho_\beta(\Phi_1(x), \Phi_2(x)). \]

Then, \( \nu_{C,\beta} \) is a pseudo-metric on \( \mathcal{M}(X, Y) \). That is, \( \nu_{C,\beta}(\Phi_1, \Phi_2) \in [0, +\infty] \) ∀\( \Phi_1, \Phi_2 \in \mathcal{M}(X, Y) \) and \( \nu_{C,\beta}(\Phi_1, \Phi_1) = 0 \) ∀\( \Phi_1 \in \mathcal{M}(X, Y) \).
Therefore,

\( \mathcal{M}(X, Y), \{\nu_{C,\beta}\}_{C,\beta} \) is a pseudo-uniform space.

We define \( \mu - \limsup \Phi_n \) to be the multifunction \( \Phi \) defined by: ∀\( x \in X \)

\[ \Phi(x) = \{y \in Y : \exists(\Phi_{n_k}) a \text{ subsequence of } (\Phi_n) \exists y_k \in \Phi_{n_k}(x) \text{ such that } y_k \xrightarrow[\mu]{} y\}. \]

Let \( (\Phi_n) \) be a sequence of elements of \( LSC_0((X, \lambda), (Y, \mu)) \). We say that \( (\Phi_n) \)
is an upper Cauchy sequence in \( (LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta}) \) if the multifunction \( \mu - \limsup \Phi_n \) is convex and

\[ \forall C \in \sigma \ \forall \beta \in I \ \nu_{C,\beta}(\Phi_n, \Phi_m) \rightarrow 0 \ \text{ as } \ n, m \rightarrow +\infty. \]

In other words, \( (\Phi_n) \) is an upper Cauchy sequence in \( (LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta}) \) if the multifunction \( \mu - \limsup \Phi_n \) is convex and ∀\( \beta \in I \ \forall \ C \in \sigma \) we have

\[ \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \ \forall m \geq n_0 \ \forall x \in C \ \forall y_1 \in \Phi_m(x) \ \exists y_2 \Phi_n(x) \ q_\beta(y_1 - y_2) < \varepsilon. \]

Let \( \zeta \) be the topology on \( \mathcal{M}(X, Y) \) generated by the family \( \{\nu_{C,\beta}\}_{C,\beta} \) of pseudo-metrics.

Recall that a sequence \( \Phi_n \) converges to \( \Phi \) with respect to the topology \( \zeta \), if

\[ \forall C \in \sigma \ \forall \beta \in I \ \nu_{C,\beta}(\Phi_n, \Phi) \rightarrow 0. \]

Let us remark that every sequence \( (\Phi_n) \) of \( \mathcal{M}(X, Y) \) converge to the multifunction \( \Phi \equiv \emptyset \) with respect to the topology \( \zeta \).

We say that \( (LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta}) \) is upper strongly sequentially complete, if every upper Cauchy sequence in \( (LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta}) \) converges to \( \mu - \limsup \Phi_n \) in \( (LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta}) \).
If \( (\Phi_n) \) is an upper Cauchy sequence in \( (\mathcal{M}(X, Y), \{\nu_{C,\beta}\}_{C,\beta}) \) converging to \( \mu - \limsup \Phi_n \) in \( (\mathcal{M}(X, Y), \{\nu_{C,\beta}\}_{C,\beta}) \), then \( \Phi \equiv \mu - \limsup \Phi_n \) is called the \( (\sigma, \mu) \)-upper multifunction limit of \( (\Phi_n) \).
For a multifunction \( \Phi : X \rightarrow Y \) and \( y' \in Y' \equiv (Y, \mu)' \), we define \( y' \circ \Phi \) to be the multifunction \( \Phi_1 : X \rightarrow R \) defined by \( \Phi_1(x) = y'(\Phi(x)) \).

Now we are ready to prove the main result of our paper.

**Theorem 2.1** \( (LSC_0((X, \lambda), R), \{\nu_{C,|\cdot|}\}_{C}) \) is upper strongly sequentially complete.
Proof. Let \((\Phi_n)\) be an upper Cauchy sequence in \(LSC_0((X, \lambda), R), \{\nu_{C,|.|}\}_C\). Then,

\[
\forall C \in \sigma \ \forall \varepsilon > 0 \ \exists n_0 \in N \ \forall n \geq n_0 \ \forall m \geq n_0 \ \forall x \in C \ \forall y_1 \in \Phi_m(x) \ \exists y_2 \in \Phi_n(x) \ |y_1 - y_2| < \frac{\varepsilon}{2}.
\]

Set \(\Phi(x) = \limsup \Phi_n(x) \equiv \sup | - \limsup \Phi_n(x) \ (\forall x)\). Let us prove that \(\Phi_n \to \Phi\) in \(LSC_0((X, \lambda), R), \{\nu_{C,|.|}\}_C\).

First full, let us prove that \(\Phi\) is the \((\sigma, |.|\))-upper multifunction limit of \((\Phi_n)\).

Let \(C \in \sigma, \beta \in I, \varepsilon > 0\). Then \(\exists n_0 \in N \ \forall n \geq n_0 \ \forall m \geq n_0 \ \forall x \in C \ \forall y_1 \in \Phi_n(x) \ \exists y_2 \in \Phi_m(x) \ |y_1 - y_2| < \frac{\varepsilon}{2}.

Let \(n \geq n_0, x \in C, y_1, y_2 \in \Phi(x)\). Since \(\Phi = \limsup \Phi_r,\) then without loss of generality, we can assume that \(\exists y_r \in \Phi_r(x) (\forall r)\) such that \(y_r \to y_1\) in \(R\).

Let \(r_0 \geq n_0\) such that \(|y_{r_0} - y_1| < \frac{\varepsilon}{2}\) and pick any \(y_2 \in \Phi_n(x)\) such that \(|y_{r_0} - y_2| < \frac{\varepsilon}{2}\). Then, we obtain \(|y_1 - y_2| < |y_{r_0} - y_2| + |y_{r_0} - y_1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\).

Consequently, \(\Phi_n \to \Phi\) in \((\mathcal{M}(X, Y), \{\nu_{C,|.|}\}_C)\). Thus, \(\Phi\) is the \((\sigma, |.|\))-upper multifunction limit of \((\Phi_n)\).

Let us prove now that \(\Phi \in LSC_0((X, \lambda), R)\). Let \(y \in \Phi(0)\) and let \(x_r \to 0\) as \(r \to +\infty\). Then, without loss of generality, we can assume that there is \(y_n \in \Phi_n(0) (\forall n)\) such that \(y_n \to y\) in \(R\). On the other hand, for each positive integer \(n, \Phi_n(0) \subset \limsup \Phi_n(x)\). Therefore, by the definition of \(\limsup \Phi_n(x)\)

and without loss of generality, we can assume that \(\exists y_n^r \in \Phi_n(x_r) (\forall n, \forall r)\) such that \(y_n^r \to y_n\) in \(R\) as \(r \to +\infty\) (\(\forall n\)). Since \(x_r \to 0\), we deduce that \(\{x_r\} \subset C(X_\lambda) \subset \sigma\). Put \(C = \{x_r\}\) and let \(\varepsilon > 0\), a simple calculation together with the above Cauchy condition shows that \(\exists n_0 \in N \ \forall n \geq n_0 \ \forall r \geq n_0 \ \exists A_n^r \in \Phi_r(x_n) \ such \ that \ |y_n^r - A_n^r| < \varepsilon\). Consequently, \(\exists (r_p)_p, \exists (n_q)_q\) two increasing sequences of positive integers such that \(|y_{n_q}^r - A_{n_q}^r| \to 0\) as \(p, q \to +\infty\). Hence, \(y_{n_q} \to y\) as \(q \to +\infty\), and \(A_{n_q}^r \to y_{n_q}\) as \(p \to +\infty\), with \(A_{n_q}^r \in \Phi_r(x_{n_q}) (\forall p, q)\). Hence, \(y \in \lim sup \Phi(x)\). On the other hand, \((\Phi_n)\)

is an upper Cauchy sequence in \(LSC_0((X, \lambda), R), \{\nu_{C,|.|}\}_C)\). Therefore, \(\Phi\) is convex. Thus, taking into account that \(\limsup \Phi_n\) is a cone, we deduce the result.

The next result is an immediate consequence of the above theorem.

Corollary 2.2 Let \((\Phi_n)\) be an upper Cauchy sequence in \(LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,|.|}\}_C\) converging to \(\Phi \equiv \mu - \limsup \Phi_n\) in \((\mathcal{M}(X, Y), \{\nu_{C,|.|}\}_C)\). Assume that \(\forall x \in X, \bigcup_n \Phi_n(x)\) is conditionally sequentially compact in \((Y, \sigma(Y, (Y, \mu)'))\).

Then, \(\Phi\) is weakly LSC at 0. That is \(\forall y' \in (Y, \mu)'\), \(y'o\Phi \in LSC_0((X, \lambda), R)\).
Proof. Let \( y' \in (Y, \mu') \). Since \((\Phi_n)\) is an upper Cauchy sequence in \((LSC_0((X, \lambda), (Y, \mu)), \nu_{C, \beta})\) and since \( \forall x \in X, \bigcup \Phi_n(x) \) is conditionally sequentially compact in \((Y, \sigma(Y, (Y, \mu')))\), then \((y' o \Phi_n)\) is an upper Cauchy sequence in \((LSC_0((X, \lambda), R), \nu_{C, ||.||})\). Therefore, using theorem 2.1, we deduce that \( y' o \Phi \in LSC_0((X, \lambda), R) \). Thus, we achieve the proof.

Remark 2.3 Let \( T_n : (X, \lambda) \rightarrow (Y, \mu) \) be a sequence of sequentially continuous linear operators satisfying \( \forall C \in \sigma \lim T_n y = Ty \) uniformly in \( y \in C \) with respect to the topology \( \mu \). Assume that the topology \( \mu \) is separated. Then the sequence of multifunctions \( \Phi_n : X \rightarrow Y \) defined by \( \Phi_n(x) = \{T_n x\} \) is an upper Cauchy sequence in \((LSC_0((X, \lambda), (Y, \mu)), \nu_{C, \beta})\) converging to \( \Phi \equiv \mu - \limsup \Phi_n = \{T\} \) in \((M(X, Y), \nu_{C, \beta})\). Moreover, \( \forall x \in X, \bigcup \Phi_n(x) \) is conditionally sequentially compact in \((Y, \sigma(Y, (Y, \mu')))\). Therefore, using corollary 2.2, we deduce that \( T \) is \((\lambda, \sigma(Y, (Y, \mu'))\) sequentially continuous. Consequently, \( T \) is \((\lambda, \mu)\) bounded. Thus, we recapture the result given by S. Lahrech in ([1], proposition 7).

References


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