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Banach-Steinhaus Type Theorems in Locally Convex Spaces for LSC Convex Processes

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Abstract

Banach-Steinhaus type results are established for LSC convex processes between locally convex spaces.

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1 Introduction

Let (X, λ) and (Y, μ) be two locally convex spaces. Assume that the locally convex topology μ is generated by the family $\{q_\beta\}_{\beta \in I}$ of semi norms on Y. Let $\beta(X_\lambda)$ denote the family of bounded sets in (X, λ) and let $\sigma \subset \beta(X_\lambda)$ such that $\bigcup_{C \in \sigma} C = X$. An operator T:$(X, \lambda) \to (Y, \mu)$ is said to be sequentially continuous if for every sequence (x_n) of X and every $x \in X$ such that $x_n \xrightarrow{\lambda} x$ one has $Tx_n \xrightarrow{\mu} Tx$.

It has been shown in [1] that if $T_n : (X, \lambda) \to (Y, \mu)$ is a sequence of sequentially continuous linear operators satisfying $\forall C \in \sigma \quad \lim T_n y = Ty$ uniformly in $y \in C$ with respect to the topology μ, then the limit operator T send λ-bounded sets into μ bounded sets. Our main objective in this paper is to generalize some results established by S. Lahrech in [1] to LSC convex processes.

Recall that a convex process $\Phi : X \to Y$ satisfying the condition: $\forall x_0 \in X$, $\Phi(x_0) \subset (\lambda, \mu)-\limsup_{x \to x_0} \Phi(x) \equiv \{y \in Y : \ \forall \ x_n \xrightarrow{\lambda} x_0, \ \exists (x_{n_k}) \text{ subsequence of } (x_n)$

$$\exists\, y_k \in \Phi(x_{n_k}) \text{ such that } \quad y_k \xrightarrow{\mu} y\}$$

is called a (λ, μ)-LSC convex process.

Assume that $C(X_\lambda) \subset \sigma$, where $C(X_\lambda)$ denote the class of sequentially relatively compacts sets in (X, λ). Denote by $LSC_0((X, \lambda), R)$ the class of LSC convex processes at 0 acting from (X, λ) into R.

2 Banach-Steinhaus theorem in locally convex spaces for LSC convex processes

Assume that all the hypotheses of the above paragraph are satisfied.

Let $\Phi : X \to Y$ be a multifunction. The domain of Φ is the set

$$D(\Phi) = \{x \in X : \Phi(x) \neq \emptyset\}.$$

We say Φ has nonempty images if its domain is X. For any subset C of X we write $\Phi(C)$ for the image $\bigcup_{x \in C} \Phi(x)$ and the range of Φ is the set $R(\Phi) = \Phi(X)$.

We say Φ is surjective if its range is Y. The graph of Φ is the set

$$G(\Phi) = \{(x, y) \in X \times Y : y \in \Phi(x)\}.$$

A multifunction is convex, or closed if its graph is likewise.

A process is a multifunction whose graph is a cone. For example, we can interpret linear closed operators as closed convex processes.

A convex process $\Phi : X \to Y$ is said to be (λ, μ) -bounded if it maps every bounded set of (X, λ) into bounded set in (Y, μ).

Denote by $\mathcal{M}(X, Y)$ the class of multifunctions acting from X into Y.

Let also $LSC_0((X, \lambda), (Y, \mu))$ denote the class of LSC convex processes at 0 from (X, λ) into (Y, μ).

For $A \subset Y$, $B \subset Y$ with $A \neq \emptyset$ and $B \neq \emptyset$, we define the direct Hausdorff distance ρ_β with respect to the semi norm q_β by: $\rho_\beta(A, B) = \sup_{y \in B} d_A^\beta(y)$, where d_A^β is the distance function to the set A with respect to the semi norm q_β. If

$A = \emptyset$ or $B = \emptyset$, then we set $\rho_\beta(A, B) = 0$.
For $\Phi_1, \Phi_2 \in \mathcal{M}(X, Y)$ and for $C \in \sigma$, $\beta \in I$, set

$$\nu_{C,\beta}(\Phi_1, \Phi_2) = \sup_{x \in C} \rho_\beta(\Phi_1(x), \Phi_2(x)).$$

Then, $\nu_{C,\beta}$ is a pseudo-metric on $\mathcal{M}(X, Y)$. That is, $\nu_{C,\beta}(\Phi_1, \Phi_2) \in [0, +\infty]$ $(\forall \Phi_1, \Phi_2 \in \mathcal{M}(X, Y))$ and $\nu_{C,\beta}(\Phi_1, \Phi_1) = 0 \ \forall \Phi_1 \in \mathcal{M}(X, Y)$.
Therefore,

$$(\mathcal{M}(X, Y), \{\nu_{C,\beta}\}_{C,\beta}) \text{ is a pseudo- uniform space.}$$

We define $\mu - \limsup \Phi_n$ to be the multifunction Φ defined by: $\forall x \in X$

$$\Phi(x) = \{y \in Y : \exists (\Phi_{n_k}) \text{ a subsequence of } (\Phi_n) \, \exists y_k \in \Phi_{n_k}(x) \text{ such that } \ y_k \stackrel{\mu}{\to} y\}.$$

Let (Φ_n) be a sequence of elements of $LSC_0((X, \lambda), (Y, \mu))$. We say that (Φ_n) is an upper Cauchy sequence in $(LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta})$ if the multifunction $\mu - \limsup \Phi_n$ is convex and

$$\forall C \in \sigma \ \forall \beta \in I \ \nu_{C,\beta}(\Phi_n, \Phi_m) \to 0 \ \text{ as } \ n, m \to +\infty.$$

In other words, (Φ_n) is an upper Cauchy sequence in $(LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta})$ if the multifunction $\mu - \limsup \Phi_n$ is convex and $\forall \beta \in I \ \forall C \in \sigma$ we have

$$\forall \varepsilon > 0 \ \exists n_0 \in N \ \forall n \geq n_0 \ \forall m \geq n_0 \ \forall x \in C \ \forall y_1 \in \Phi_m(x) \ \exists \, y_2 \Phi_n(x) \ q_\beta(y_1 - y_2) < \varepsilon.$$

Let ζ be the topology on $\mathcal{M}(X, Y)$ generated by the family $\{\nu_{C,\beta}\}_{C,\beta}$ of pseudo-metrics.
Recall that a sequence Φ_n converges to Φ with respect to the topology ζ, if

$$\forall C \in \sigma \ \forall \beta \in I \ \nu_{C,\beta}(\Phi_n, \Phi) \to 0.$$

Let us remark that every sequence (Φ_n) of $\mathcal{M}(X, Y)$ converge to the multifunction $\Phi \equiv \emptyset$ with respect to the topology ζ.
We say that $(LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta})$ is upper strongly sequentially complete, if every upper Cauchy sequence in $(LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta})$ converges to $\mu - \limsup \Phi_n$ in $(LSC_0((X, \lambda), (Y, \mu)), \{\nu_{C,\beta}\}_{C,\beta})$.
If (Φ_n) is an upper Cauchy sequence in $(\mathcal{M}(X, Y), \{\nu_{C,\beta}\}_{C,\beta})$ converging to $\mu - \limsup \ \Phi_n$ in $(\mathcal{M}(X, Y), \{\nu_{C,\beta}\}_{C,\beta})$, then $\Phi \equiv \mu - \limsup \ \Phi_n$ is called the (σ, μ)-upper multifunction limit of (Φ_n).
For a multifunction $\Phi : X \to Y$ and $y' \in Y' \equiv (Y, \mu)'$, we define $y' o \Phi$ to be the multifunction $\Phi_1 : X \to R$ defined by $\Phi_1(x) = y'(\Phi(x))$.
Now we are ready to prove the main result of our paper.

Theorem 2.1 $(LSC_0((X, \lambda), R), \{\nu_{C,|.|}\}_C)$ *is upper strongly sequentially complete.*

Proof. Let (Φ_n) be an upper Cauchy sequence in $(LSC_0((X,\lambda),R),\{\nu_{C,|.|}\}_C)$. Then,

$$\forall C\in\sigma\ \forall\varepsilon>0\ \exists\, n_0\in N\ \forall\, n\geq n_0\,\forall\, m\geq n_0\ \forall\, x\in C\ \forall\, y_1\in\Phi_m(x)\ \exists\, y_2\in\Phi_n(x)$$

$$\mid y_1-y_2\mid<\frac{\varepsilon}{2}.$$

Set $\Phi(x)=\limsup\ \Phi_n(x)\equiv\mid.\mid-\limsup\ \Phi_n(x)\ (\forall x)$. Let us prove that

$$\Phi_n\rightarrow\Phi \text{ in } (LSC_0((X,\lambda),R),\{\nu_{C,|.|}\}_C).$$

First full, let us prove that Φ is the $(\sigma,\mid.\mid)$-upper multifunction limit of (Φ_n). Let $C\in\sigma$, $\beta\in I$, $\varepsilon>0$. Then $\exists\, n_0\in N\ \forall\, n\geq n_0\,\forall\, m\geq n_0\ \forall\, x\in C\ \forall\, y_1\in\Phi_n(x)\ \exists\, y_2\in\Phi_m(x)\ \ \mid y_1-y_2\mid<\frac{\varepsilon}{2}$.
Let $n\geq n_0$, $x\in C$, $y_1\in\Phi(x)$. Since $\Phi=\limsup\Phi_r$, then without loss of generality, we can assume that $\exists\, y_r\in\Phi_r(x)\,(\forall\, r)$ such that $y_r\rightarrow y_1$ in R.
Let $r_0\geq n_0$ such that $\mid y_{r_0}-y_1\mid<\frac{\varepsilon}{2}$ and pick any $y_2\in\Phi_n(x)$ such that $\mid y_{r_0}-y_2\mid<\frac{\varepsilon}{2}$. Then, we obtain $\mid y_1-y_2\mid<\mid y_1-y_{r_0}\mid+\mid y_{r_0}-y_2\mid<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Consequently, $\Phi_n\rightarrow\Phi$ in $(\mathcal{M}(X,Y),\{\nu_{C,\beta}\}_{C,\beta})$. Thus, Φ is the $(\sigma,\mid.\mid)$-upper multifunction limit of (Φ_n).

Let us prove now that $\Phi\in LSC_0((X,\lambda),R)$. Let $y\in\Phi(0)$ and let $x_r\overset{\lambda}{\rightarrow}0$ as $r\rightarrow+\infty$. Then, without loss of generality, we can assume that there is $y_n\in\Phi_n(0)$ $(\forall\, n)$ such that $y_n\rightarrow y$ in R. On the other hand, for each positive integer n, $\Phi_n(0)\subset\limsup\limits_{x\overset{\lambda}{\rightarrow}0}\Phi_n(x)$. Therefore, by the definition of $\limsup\limits_{x\rightarrow 0}\Phi_n(x)$ and without loss of generality, we can assume that $\exists\, y_n^r\in\Phi_n(x_r)$ $(\forall\, n,\forall\, r)$ such that $y_n^r\rightarrow y_n$ in R as $r\rightarrow+\infty$ $(\forall\, n)$. Since $x_r\overset{\lambda}{\rightarrow}0$, we deduce that $\{x_r\}\in C(X_\lambda)\subset\sigma$. Put $C=\{x_r\}$ and let $\varepsilon>0$, a simple calculation together with the above Cauchy condition shows that $\exists\, n_0\in N\ \forall\, n\geq n_0\,\forall\, r\geq n_0\ \exists\, A_n^r\in\Phi_r(x_n)$ such that $\mid y_n^r-A_n^r\mid<\varepsilon$. Consequently, $\exists\,(r_p)_p$, $\exists\,(n_q)_q$ two increasing sequences of positive integers such that $\mid y_{n_q}^{r_p}-A_{n_q}^{r_p}\mid\rightarrow 0$ as $p,q\rightarrow+\infty$. Hence, $y_{n_q}\rightarrow y$ as $q\rightarrow+\infty$, and $A_{n_q}^{r_p}\rightarrow y_{n_q}$ as $p\rightarrow+\infty$, with $A_{n_q}^{r_p}\in\Phi_{r_p}(x_{n_q})$ $(\forall p,q)$. Hence, $y\in\limsup\limits_{x\overset{\lambda}{\rightarrow}0}\Phi(x)$. On the other hand, (Φ_n) is an upper Cauchy sequence in $(LSC_0((X,\lambda),R),\{\nu_{C,|.|}\}_C)$. Therefore, Φ is convex. Thus, taking into account that $\limsup\Phi_n$ is a cone, we deduce the result.
The next result is an immediate consequence of the above theorem.

Corollary 2.2 *Let (Φ_n) be an upper Cauchy sequence in $(LSC_0((X,\lambda),(Y,\mu)),$ $\{\nu_{C,\beta}\}_{C,\beta})$ converging to $\Phi\equiv\mu-\limsup\Phi_n$ in $(\mathcal{M}(X,Y),\{\nu_{C,\beta}\}_{C,\beta})$. Assume that $\forall x\in X$, $\bigcup\limits_n\Phi_n(x)$ is conditionally sequentially compact in $(Y,\sigma(Y,(Y,\mu)'))$. Then, Φ is weakly LSC at 0. That is $\forall\, y'\in(Y,\mu)'$, $y'o\Phi\in LSC_0((X,\lambda),R)$.*

Proof. Let $y' \in (Y,\mu)'$. Since (Φ_n) is an upper Cauchy sequence in $(LSC_0((X,\lambda),(Y,\mu)),\{\nu_{C,\beta}\}_{C,\beta})$ and since $\forall x \in X$, $\bigcup_n \Phi_n(x)$ is conditionally sequentially compact in $(Y,\sigma(Y,(Y,\mu)'))$, then $(y'o\Phi_n)$ is an upper Cauchy sequence in $(LSC_0((X,\lambda),R),\{\nu_{C,|.|}\}_C)$. Therefore, using theorem 2.1, we deduce that $y'o\Phi \in LSC_0((X,\lambda),R)$. Thus, we achieve the proof.

Remark 2.3 *Let $T_n : (X,\lambda) \to (Y,\mu)$ be a sequence of sequentially continuous linear operators satisfying $\forall C \in \sigma$ $\lim T_n y = Ty$ uniformly in $y \in C$ with respect to the topology μ. Assume that the topology μ is separated. Then the sequence of multifunctions $\Phi_n : X \to Y$ defined by $\Phi_n(x) = \{T_n x\}$ is an upper Cauchy sequence in $(LSC_0((X,\lambda),(Y,\mu)),\{\nu_{C,\beta}\}_{C,\beta})$ converging to $\Phi \equiv \mu-\limsup\ \Phi_n = \{T\}$ in $(\mathcal{M}(X,Y),\{\nu_{C,\beta}\}_{C,\beta})$. Moreover, $\forall x \in X$, $\bigcup_n \Phi_n(x)$ is conditionally sequentially compact in $(Y,\sigma(Y,(Y,\mu)'))$. Therefore, using corollary 2.2, we deduce that T is $(\lambda,\sigma(Y,(Y,\mu)')$ sequentially continuous. Consequently, T is (λ,μ) bounded. Thus, we recapture the result given by S. Lahrech in ([1], proposition 7).*

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