Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 22, 1053 - 1067

A Self Adjoint Expansion of a Symmetric Differential Operator with Operator Coefficient

Ehliman Adigüzelov and Serpil Karayel[1]

Department of Mathematics
Faculty of Arts and Science
Yıldız Technical University
(34210), Davutpaşa, İstanbul, Turkey

Abstract

In this work, we prove that the closure of a symmetric operator L_0 which is formed by differential expression

$$(L_0 y)(x) = -(p(x)y'(x))' - Q(x)y(x)$$

and with the boundary condition

$$\cos\alpha . y(0) + \sin\alpha . y'(0) = 0$$

is self adjoint where $\alpha \in (-\infty, \infty)$ in the space $L_2(0, \infty; H)$. Furthermore, we investigate some properties of this operator.

Mathematics Subject Classification: 47A10, 34L20

Keywords: Closure, Hilbert Space, Self-Adjoint Operator, Symmetric Operator

1 INTRODUCTION

Let H be a separable Hilbert space with infinite dimension. We will denote the inner products with $(.,.)$ and $(.,.)_{(0,\infty)}$ in the spaces H and $L_2(0,\infty; H)$ respectively. Let $Q(x)$ be self adjoint operator from H to H for all x in

[1]serpil@yildiz.edu.tr

$[0,\infty)$. Moreover, we consider $Q(x)$ as a continuous operator function in the interval $[0,\infty)$ with respect to the norm on the space $B(H)$. We suppose that there are positive constants c_1, c_2 such that

$$c_1 \leq p(x) \leq c_2.$$

Let $D(L_0)$ be set of all functions $y(x)$ which satisfy the following conditions in the interval $[0,\infty)$;

1) $y(x)$ has continuous second order derivative with respect to the norm on the space H.

2) $\cos\alpha . y(0) + \sin\alpha . y'(0) = 0$ (1.1)

where $\alpha \in (-\infty,\infty)$ is a constant.

3) $y(x)$ has compact support at infinity.

We consider the operator $L_0 : D(L_0) \to L_2(0,\infty;\ H)$ defined by

$$(L_0 y)(x) = -(p(x)y'(x))' - Q(x)y(x). \tag{1.2}$$

We have $\overline{D(L_0)} = L_2(0,\infty;\ H)$. For all $y, z \in D(L_0)$, there exists a number $b \in (0,\infty)$ which depends on y and z such that

$$y^{(i)}(x) = z^{(i)}(x) = 0 \qquad (i = 0, 1, 2)$$

when $x \geq b$. By using integration by parts, we obtain

$$\begin{aligned}
(L_0 y,\ z)_{(0,\infty)} &= \int_0^\infty (-(p(x)y'(x))' - Q(x)y(x),\ z(x))dx \\
&= -(p(x)y'(x),\ z(x))\Big|_0^b + \int_0^b (p(x)y'(x), z'(x))dx \\
&\qquad - \int_0^b (y(x), Q(x)z(x))dx \\
&= p(0)(y'(0), z(0)) + (y(x), p(x)z'(x))\Big|_0^b \\
&- \int_0^b (y(x), (p(x)z'(x))')dx - \int_0^b (y(x), Q(x)z(x))dx \\
&= p(0)(y'(0), z(0)) - p(0)(y(0), z'(0)) \\
&\quad + \int_0^b (y(x), -(p(x)z'(x))' - Q(x)z(x))dx
\end{aligned} \tag{1.3}$$

Let us show that

$$(y'(0),\ z(0)) - (y(0),\ z'(0)) = 0. \tag{1.4}$$

If $\cos\alpha = 0$, the boundary condition (1.1) is transformed to the condition $y'(0) = 0$. In this case, it is obvious that the equality (1.4) is satisfied. If $\cos\alpha \neq 0$, then we have

$$y(0) = -y'(0)\tan\alpha$$
$$z(0) = -z'(0)\tan\alpha.$$

Therefore, we have

$$(y'(0), z(0)) - (y(0), z'(0)) = -\tan\alpha.(y'(0), z'(0)) + \tan\alpha.(y'(0), z'(0)) = 0$$

It can be easily seen that the equality (1.4) is satisfied for all fixed number $\alpha \in (-\infty, \infty)$. From (1.3) and (1.4), we find

$$(L_0 y,\ z)_{(0,\infty)} = (y,\ L_0 z)_{(0,\infty)}.$$

Thus, we have showed that L_0 is a symmetric operator. So, the operator L_0 has a closure $\overline{L_0}$. In this work, we have proved that the operator $\overline{L_0}$ is self adjoint under some conditions and we have investigated some properties of this operator.

Let $f : [a, b] \to H$ be any function. If h is an arbitrary element of the space H and $g : [a, b] \to H$ is an integrable, in the sense of Bochner, function on the interval $[a, b]$ such that

$$f(x) = h + \int_a^x g(t)dt,$$

then f is said to be an absolutely continuous function on the interval $[a, b]$. Here, the Bochner integral of the function g has been shown by $\int_a^x g(t)dt$ in the interval $[a, x]$.

In the works [1], [2], [3], [4], [5] and [6], the self adjoint expansions of the closures of symmetric operators formed by some differential expressions have been investigated.

By the work [2], it is known that the closure of every symmetric operator which has a complete set of eigenvectors is self adjoint.

2 ABOUT THE ADJOINT OPERATOR L_0^* OF THE OPERATOR L_0

In this part, we will investigate some properties of the operator L_0^*.

Lemma 2.1 *Let $f(x)$ $(a \le x \le b)$ be a function whose values belong to the space H and satisfies the following conditions;*

a) $f(x)$ has absolutely continuous derivative with respect to the norm on the space H in the interval $[a,b]$.

b) $f''(x) \in L_2(a,b;H)$.

c) $f(a) = f'(a) = f(b) = f'(b) = 0$

If $h(x) \in L_2(a,b;H)$ is a function satisfying the equality

$$\int_a^b (f''(x), h(x))dx = 0$$

then $h(x) = g + xe$ for all $f(x)$ satisfying the conditions a), b) and c). Here, g and e are fixed elements of H.

Let $D(L_0)$ be the set of all functions $y(t)$ which have compact support in $[0,\infty)$ and continuous second order derivatives with respect to the norm on the space H and satisfying the condition (1.1). Moreover, let A and B define as follows;

$$A = \{y(0) : \ y \in D(L_0)\}$$
$$B = \{y'(0) : \ y \in D(L_0)\}.$$

It can be easily proved that if $\sin\alpha \neq 0$ then $A = H$ and if $\cos\alpha \neq 0$ then $B = H$ respectively.

Let $D(L_1)$ denote the set of all functions $y(t) \in L_2(0,\infty;H)$ satisfying the following conditions;

1) $y(t)$ has continuous second order derivative with respect to the norm on the space H.

2) $y(t)$ has compact support at infinity.

3) $y(0) = y'(0) = 0$.

Let us consider the operator $L_1 : D(L_1) \to L_2(0,\infty;\ H)$ defined by

$$(L_1y)(x) = -(p(x)y'(x))' - Q(x)y(x).$$

First of all, by using lemma 2.1, let us investigate the operator L_1^*.

Theorem 2.2 *Every function* $z = z(x) \in D(L_1^*)$ *satisfies the following conditions;*

a) $z(x)$ *and* $z'(x)$ *are absolutely continuous with respect to the norm on the space* H *in every finite interval* $[0, a]$.

b) $-(p(x)z'(x))' - Q(x)z(x) \in L_2(0, \infty;\ H)$,

and we have

$$(L_1^* z)(x) = -(p(x)z'(x))' - Q(x)z(x).$$

Proof: By the definition of adjoint operator, if $z = z(x) \in D(L_1^*)$, there exists a function $z^* = z^*(x) \in L_2(0, \infty;\ H)$ such that

$$(L_1 y,\ z)_{(0,\infty)} = (y,\ z^*), \qquad (y \in D(L_1)). \tag{2.1}$$

The equality (2.1) can be written as the form

$$\int_0^\infty (-(p(x)y'(x))' - Q(x)y'(x), z(x))dx = \int_0^\infty (y(x), z^*(x))dx$$

or

$$-\int_0^\infty ((p(x)y'(x))', z(x))dx = \int_0^\infty (y(x), u(x))dx \tag{2.2}$$

where $u(x) = Q(x)z(x) + z^*(x)$. We consider that there exists a constant $b \in (0, \infty)$ such that $y(x) = y'(x) = 0$ for all $x \geq b$. By using this consideration and integration by parts, we obtain

$$\begin{aligned}
\int_0^\infty ((p(x)y'(x))', z(x))dx &= \int_0^b (p'(x)y'(x), z(x))dx + \int_0^b (p(x)y''(x), z(x))dx \\
&= \int_0^b \Big(y'(x), (\int_0^x p'(t)z(t)dt)'\Big)dx + \int_0^b (y''(x), p'(x)z(x))dx \\
&= \Big(y'(x), \int_0^x p'(t)z(t)dt\Big)\Big|_0^b - \int_0^b \Big(y''(x), \int_0^x p'(t)z(t)dt\Big)dx \\
&\quad + \int_0^b (y''(x), p(x)z(x))dx
\end{aligned}$$

$$= \int_0^\infty \Big(y''(x), p(x)z(x) - \int_0^x p'(t)z(t)dt\Big)dx \tag{2.3}$$

$$\begin{aligned}
\int_0^\infty (y(x), u(x))dx &= \int_0^b \Big(y(x), (\int_0^x u(t)dt)'\Big)dx \\
&= \Big(y(x), \int_0^x u(t)dt\Big)\Big|_0^b - \int_0^b \Big(y'(x), \int_0^x u(t)dt\Big)dx \\
&= -\int_0^b \Big(y'(x), \int_0^x (\int_0^t u(\tau)d\tau)'dt\Big)dx \\
&= -\Big(y'(x), \int_0^x (\int_0^t u(\tau)d\tau)dt\Big)\Big|_0^b + \int_0^b \Big(y''(x), \int_0^x (\int_0^t u(\tau)d\tau)dt\Big)dx \\
&= \int_0^b \Big(y''(x), t\int_0^t u(\tau)d\tau\Big|_0^x - \int_0^x tu(t)dt\Big)dx
\end{aligned}$$

$$= \int_0^\infty \Big(y''(x), \int_0^x (x-t)u(t)dt\Big)dx \tag{2.4}$$

From (2.2), (2.3) and (2.4), we obtain

$$-\int_0^\infty \Big(y''(x), p(x)z(x) - \int_0^x p'(t)z(t)dt\Big)dx = \int_0^\infty \Big(y''(x), \int_0^x (x-t)u(t)dt\Big)dx$$

or

$$\int_0^\infty \Big(y''(x), -p(x)z(x) + \int_0^x p'(t)z(t)dt - \int_0^x (x-t)u(t)dt\Big)dx = 0 \tag{2.5}$$

Since the equality (2.5) is satisfied for all functions $y(x) \in D(L_1)$, by the lemma 2.1, we have

$$-p(x)z(x) + \int_0^x p'(t)z(t)dt - \int_0^x (x-t)u(t)dt = g + xe. \tag{2.6}$$

Here, $g \in H$ and $e \in H$ are elements which depend on the function $z = z(x) \in D(L_1^*)$. Seen as from (2.6), $z(x)$ is absolutely continuous in every finite interval $[0, a]$.
If we take derivatives of the equation (2.6) with respect to x then, we obtain

$$-p'(x)z(x) - p(x)z'(x) + p'(x)z(x) - \int_0^x u(t)dt = e. \tag{2.7}$$

From here, it is seen that the function $z'(x)$ is also absolutely continuous with respect to the norm on the space H in every finite interval. From (2.7), we find

$$-(p(x)z'(x))' - u(x) = 0.$$

If we consider the following equality

$$u(x) = Q(x)z(x) + z^*(x)$$

then, we find

$$-(p(x)z'(x))' - Q(x)z(x) = z^*(x).$$

Therefore, we obtain

$$-(p(x)z'(x))' - Q(x)z(x) \in L_2(0, \infty;\ H)$$

and

$$(L_1^* z)(x) = -(p(x)z'(x))' - Q(x)z(x). \quad \square$$

Theorem 2.3 *The set* $D(L_0^*)$ *consists of all functions* $z = z(x)$ *of the space* $L_2(0, \infty;\ H)$ *satisfying the following conditions;*

a) $z(x)$ *and* $z'(x)$ *are absolutely continuous with respect to the norm on the space* H *in every finite interval* $[0, a]$ $\quad (a \in (0, \infty))$.

b) $-(p(x)z'(x))' - Q(x)z(x) \in L_2(0, \infty;\ H)$.

c) $\cos\alpha.z(0) + \sin\alpha.z'(0) = 0$.

Moreover, we have

$$(L_0^* z)(x) = -(p(x)z'(x))' - Q(x)z(x).$$

Proof: Since $L_1 \subset L_0$, we have $L_0^* \subset L_1^*$. Thus, by the Theorem 2.2, every function $z = z(x) \in D(L_0^*)$ satisfies the conditions a), b) and also we have

$$(L_0^* z)(x) = (L_1^* z)(x) = -(p(x)z'(x))' - Q(x)z(x). \tag{2.8}$$

Let us show that the condition c) is satisfied for every $z \in D(L_0^*)$. By the definition of adjoint operator, if $z \in D(L_0^*)$, we have

$$(L_0 y, z)_{(0,\infty)} = -\int_0^\infty ((p(x)y'(x))' + Q(x)y(x), z(x))dx$$

$$= \int_0^\infty (y(x), z^*(x))dx = (y, z^*)_{(0,\infty)} \quad (y \in D(L_0)) \tag{2.9}$$

By using integration by parts, we obtain

$$-\int_0^\infty ((p(x)y'(x))', z(x))dx = p(0)(y'(0), z(0)) - p(0)(y(0), z'(0))$$

$$-\int_0^\infty (y(x), (p(x)z'(x))')dx \tag{2.10}$$

From (2.9) and (2.10), we find

$$p(0)(y'(0), z(0)) - p(0)(y(0), z'(0)) + \int_0^\infty (y(x), -(p(x)z'(x))' - Q(x)z(x))dx$$

$$= (y, L_0^* z)_{(0,\infty)}. \tag{2.11}$$

From (2.8) and (2.11), we obtain

$$(y'(0), z(0)) - (y(0), z'(0)) = 0, \qquad (y \in D(L_0)). \tag{2.12}$$

If $\sin\alpha \neq 0$, from the boundary condition (1.1), we find

$$y'(0) = -\cot\alpha . y(0).$$

If we replace the last expression in (2.12) then, we obtain

$$(-\cot\alpha . y(0), z(0)) - (y(0), z'(0)) = 0$$

or

$$(y(0), \cos\alpha . z(0) + \sin\alpha . z'(0)) = 0, \qquad (y \in D(L_0)).$$

Assuming $A = \{y(0) : y \in D(L_0)\} = H$ when $\sin\alpha \neq 0$, from the last relation it follows that

$$\cos\alpha . z(0) + \sin\alpha . z'(0) = 0.$$

If $\cos\alpha \neq 0$, from the boundary condition (1.1), we obtain

$$y(0) = -\tan\alpha . y'(0).$$

If this expression is replaced in (2.12) then, we obtain

$$(y'(0), z(0)) + (\tan\alpha . y'(0), z'(0)) = 0$$

or

$$(y'(0), \cos\alpha . z(0) + \sin\alpha . z'(0)) = 0, \qquad (y \in D(L_0)).$$

Assuming $B = \{y'(0) : (y \in D(L_0))\} = H$ when $\cos\alpha \neq 0$, from the last relation it follows that

$$\cos\alpha . z(0) + \sin\alpha . z'(0) = 0.$$

Finally, let us show that every function $z(x) \in L_2(0, \infty; H)$ satisfying the conditions a), b) and c), belongs to the set $D(L_0^*)$ and satisfies the following equality;

$$(L_0^* z)(x) = -(p(x) z'(x))' - Q(x) z(x).$$

Let $y \in D(L_0)$ and let z be any function which satisfies the conditions a), b) and c). There exists a number $b \in (0, \infty)$ such that $y(x) = y'(x) = 0$ for every $x \geq b$. Hence, by using integration by parts, we obtain

$$\int_0^\infty ((p(x) y'(x))', z(x)) dx = p(0)(y(0), z'(0)) - p(0)(y'(0), z(0))$$

$$+\int_0^\infty (y(x),(p(x)z'(x))')dx. \tag{2.13}$$

If $\cos\alpha \neq 0$, in this case, from the conditions

$$\cos\alpha.y(0)+\sin\alpha.y'(0)=0$$

$$\cos\alpha.z(0)+\sin\alpha.z'(0)=0$$

we find

$$y(0)=-\tan\alpha.y'(0)$$

$$z(0)=-\tan\alpha.z'(0).$$

Therefore, we obtain

$$(y(0),z'(0))-(y'(0),z(0))=-\tan\alpha(y'(0),z'(0))+\tan\alpha(y'(0),z'(0))=0 \tag{2.14}$$

From (2.13) and (2.14), we find

$$\int_0^\infty ((p(x)y'(x))',z(x))dx=\int_0^\infty (y(x),(p(x)z'(x))')dx \tag{2.15}$$

By using (2.15), we obtain

$$(L_0y,\ z)_{(0,\infty)}=\int_0^\infty (y(x),-(p(x)z'(x))'-Q(x)z(x))dx=(y,L_0^*z)_{(0,\infty)}$$

Consequently, if the function $z=z(x)\in L_2(0,\infty;H)$ satisfies the conditions a), b) and c) then, we have $z\in D(L_0^*)$ and

$$(L_0^*z)(x)=-(p(x)z'(x))'-Q(x)z(x).\Box$$

3 PROPERTIES OF THE OPERATOR $\overline{L_0}$

In this section, we will prove that the operator $L=\overline{L_0}$ is self adjoint.

Lemma 3.1 *Let us suppose that for all* $x \in [0,\infty)$ *there exists a constant* $\beta \in R$ *such that* $Q(x) \leq \beta I$. *If the function* $y(x) \in L_2(0,\infty;H)$ *satisfies the following conditions;*

a) The functions $y(x)$ *and* $y'(x)$ *are absolutely continuous with respect to the norm on the space* H *in every finite interval* $[0,a]$, $\quad (a \in (0,\infty))$

b) $\ell(y) = -(p(x)y'(x))' - Q(x)y(x) \in L_2(0,\infty;\ H)$

then, we have $y'(x) \in L_2(0,\infty;H)$.

Proof: By using integration by parts, we obtain

$$\int_0^a ((p(x)y'(x))', y(x))dx = (p(x)y'(x), y(x))\Big|_0^a - \int_0^a p(x)\|y'(x)\|^2 dx$$

or

$$\int_0^a ((p(x)y'(x))', y(x))dx + \int_0^a p(x)\|y'(x)\|^2 dx = p(a)(y'(a), y(a)) - p(0)(y'(0), y(0)) \tag{3.1}$$

for all $a \in (0,\infty)$.

Since $y \in L_2(0,\infty;H)$ and $\ell(y) \in L_2(0,\infty;H)$, we obtain existence of the finite limit;

$$\lim_{a\to\infty}\Big[\int_0^a ((p(x)y'(x))', y(x))dx + \int_0^a (Q(x)y(x), y(x))dx\Big]. \tag{3.2}$$

We have

$$\int_0^a (Q(x)y(x), y(x))dx = \int_0^a ((Q(x)-\beta I)y(x), y(x))dx + \beta\int_0^a \|y(x)\|^2 dx \tag{3.3}$$

Since $((Q(x)-\beta I)y(x), y(x)) \leq 0$, there exists the limit;

$$\lim_{a\to\infty}\int_0^a ((Q(x)-\beta I)y(x), y(x))dx. \tag{3.4}$$

This limit is either finite or $-\infty$. If we consider $y(x) \in L_2(0,\infty;H)$, namely, existence of the finite limit;

$$\lim_{a\to\infty}\int_0^a \|y(x)\|^2dx,$$

from (3.3), we obtain existence of the limit;

$$\lim_{a\to\infty}\int_0^a (Q(x)y(x), y(x))dx. \tag{3.5}$$

The limit (3.5) is finite or $-\infty$ depending on the limit (3.4) being either finite or $-\infty$, respectively. Thus, from the existence of the finite limit (3.2), we obtain existence of the finite limit;

$$\lim_{a\to\infty}\int_0^a ((p(x)y'(x))', y(x))dx \tag{3.6}$$

or

$$\lim_{a\to\infty} Re\int_0^a ((p(x)y'(x))', y(x))dx = \infty. \tag{3.7}$$

Let us show the existence of the finite limit (3.6). Suppose that the limit (3.6) does not exists. Thus, the equality (3.7) is satisfied. From (3.1) and (3.7), we find

$$\lim_{a\to\infty} Re(y'(a), y(a)) = \infty \tag{3.8}$$

On the other hand, we have

$$(\|y(a)\|^2)' = 2Re(y'(a), y(a)). \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\lim_{a\to\infty}(\|y(a)\|^2)' = \infty.$$

Then, the function $\|y(x)\|^2$ is monotonous increasing in any interval $[b,\infty)$, $(b\in(0,\infty))$. This result contradicts with $y(x)\in L_2(0,\infty;H)$. Consequently, the finite limit (3.6) exists. Hence, from (3.1), we find

$$\lim_{a\to\infty}\int_0^a p(x)\|y'(x)\|^2dx = \int_0^\infty p(x)\|y'(x)\|^2dx < \infty. \tag{3.10}$$

If we consider $p(x)\ge c_1>0$, from (3.10), we obtain

$$\int_0^\infty \|y'(x)\|^2dx < \infty$$

namely, $y'(x)\in L_2(0,\infty;\ H)$.

Theorem 3.2 *If for all $x \in [0, \infty)$ there exists a constant $\beta \in (-\infty, \infty)$ which depends on x such that $Q(x) \leq \beta I$ then, the operator $L = \overline{L_0}$ is self adjoint and $D(L)$ consists of all functions $y = y(x)$ of the space $L_2(0, \infty; H)$ satisfying the following conditions;*

a) $y(x)$ and $y'(x)$ are absolutely continuous with respect to the norm on the space H in every finite interval $[0, a]$, $(a \in (0, \infty))$.

b) $-(p(x)y'(x))' - Q(x)y(x) \in L_2(0, \infty;\ H)$.

c) $\cos\alpha.y(0) + \sin\alpha.y'(0) = 0$.

Furthermore, we have

$$(Ly)(x) = -(p(x)y'(x))' - Q(x)y(x), \qquad (y \in D(L)).$$

Proof: Let y and z be any two elements of the set $D(L_0^*)$. Since $y(x), z(x) \in L_2(0, \infty;\ H)$, by the Theorem 2.3, we have

$$-(p(x)y'(x))' - Q(x)y(x), \quad -(p(x)z'(x))' - Q(x)z(x) \in L_2(0, \infty;\ H).$$

Thus, by the Lemma 3.1, we have $y'(x), z'(x) \in L_2(0, \infty;\ H)$.
So, we have $(y'(x), z(x))$, $\quad (y(x), z'(x)) \in L_1[0, \infty)$.
Therefore, there exists a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = \infty$$

and

$$\lim_{n\to\infty} [(y'(a_n), z(a_n)) - (y(a_n), z'(a_n))] = 0 \tag{3.11}$$

For this sequence $\{a_n\}$, we have

$$(L_0^* y,\ z)_{(0,\infty)} = \int_0^\infty (-(p(x)y'(x))' - Q(x)y(x), z(x))dx$$

$$= \lim_{n\to\infty} \int_0^{a_n} -((p(x)y'(x))', z(x))dx - \int_0^{a_n} (Q(x)y(x), z(x))dx. \tag{3.12}$$

Moreover, by using the conditions

$$\cos\alpha.y(0) + \sin\alpha.y'(0) = \cos\alpha.z(0) + \sin\alpha.z'(0) = 0$$

and integration by parts, we obtain

$$\int_0^{a_n} ((p(x)y'(x))', z(x))dx = p(a_n)[(y'(a_n), z(a_n)) - (y(a_n), z'(a_n))]$$

$$+ \int_0^{a_n} (y(x), (p(x)z'(x))')dx \qquad (3.13)$$

From (3.11), (3.12) and (3.13), we find

$$(L_0^* y,\ z)_{(0,\infty)} = \int_0^{\infty} (y(x), -(p(x)z'(x))' - Q(x)z(x))dx = (y,\ L_0^* z)_{(0,\infty)}.$$

It is seen as, L_0^* is a symmetric operator. Therefore, we have

$$L_0^* \subset (L_0^*)^*. \qquad (3.14)$$

Since L_0 is also a symmetric operator, we have $L_0 \subset L_0^*$. In this case, it is known that

$$(L_0^*)^* \subset L_0^*. \qquad (3.15)$$

From (3.14) and (3.15), we obtain

$$L_0^* = (L_0^*)^*. \qquad (3.16)$$

Namely, the operator L_0^* is self adjoint. On the other hand, it is known that

$$(L_0^*)^* = \overline{L_0}. \qquad (3.17)$$

From (3.16) and (3.17), we find

$$\overline{L_0} = L_0^*.$$

Hence, the operator $L = \overline{L_0} = L_0^*$ is self adjoint and the set $D(L) = D(\overline{L_0})$ consists of the functions $y(x)$ satisfying the conditions a), b) and c) in the space $L_2(0,\infty;\ H)$ by the Theorem 2.3 . Moreover, we have

$$(Ly)(x) = (\overline{L_0}y)(x) = -(p(x)y'(x))' - Q(x)y(x).$$

References

[1] A.A. Abudov, M. Bayramoglu, Essential self adjointness of Sturm-Liouville operator with operator coefficient, Spectral Theory of Operators,Baku Elm, (1982),12-20.

[2] E.E. Adiguzelov, Ö. Bakşi, On the regularized trace of the differential operator equation given in a finite interval, Journal of Engineering and Natural Sciences Sigma, (2004/1),47-55.

[3] R.K. Amirov, I.M. Guseinov, Boundary value problems for a class of Sturm-Liouville operators with nonintegrable potensial, Differential Equations, 38(2002), No 8, 1195-1197.

[4] M. Bairamoglu, About self adjoint expansions of an operator equation which has singularity, Izv. AN Azerb. SSR, ser.fiz.-teknn.i mat. nauk, 2(1976), 140-144.

[5] M. Bairamoglu, About essential self adjointness of Sturm-Liouville operator equation, Dokl.AN Azerb. SSR, (1977), XXXIII, No:5.

[6] M. Otelbaev, On the conditions of self adjointness of Schrödinger operator with operator coefficient, Ukr. Math. Journal, 28(1976), No:6, 763-771.

Received: March 1, 2007