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A Subclass of Harmonic Meromorphic Functions

Aini Janteng

School of Science and Technology
Universiti Malaysia Sabah
Locked Bag No. 2073
88999 Kota Kinabalu, Sabah, Malaysia
aini_jg@ums.edu.my

Suzeini Abdul Halim

Institute of Mathematical Sciences
Universiti Malaya
50603 Kuala Lumpur, Malaysia
suzeini@um.edu.my

Abstract

Complex-valued harmonic meromorphic functions that are univalent and orientation preserving outside the unit circle $\widetilde{U}$ can be written in the form $f = h + \overline{g}$, where h and g are analytic in $\widetilde{U}$. We define and investigate a subclass of harmonic meromorphic functions. We obtain coefficient conditions, extreme points, distortion bounds, convolution conditions and convex combinations for the above subclass of harmonic meromorphic functions.

Keywords: harmonic, meromorphic, univalent

1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathcal{D} \subseteq \mathcal{C}$ if both u and v are real harmonic in $\mathcal{D}$. Hengartner and Schober [1], among other things, investigated the family $\sum_{\mathcal{H}}$ of functions $f = h + \overline{g}$ which are harmonic, meromorphic, orientation preserving and univalent in $\widetilde{U} = \{z : |z| > 1\}$ where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n}, \quad a_n \geq 0,\ b_n \geq 0,\ z \in \widetilde{U}. \qquad (1)$$

Motivated by the results of [1], Jahangiri and Silverman [3] and Jahangiri [2] studied the classes of functions in $\sum_{\mathcal{H}}$ which are starlike or convex in $\widetilde{U}$. Also let $\sum_{\bar{\mathcal{H}}}$ consisting of functions $f = h + \overline{g}$ where h and g are of the form

$$h(z) = z + \sum_{n=1}^{\infty} |a_n| z^{-n}, \quad g(z) = -\sum_{n=1}^{\infty} |b_n| z^{-n}, \quad z \in \widetilde{U}. \tag{2}$$

Now, we consider the class of functions as follows:

Definition 1.1 *Let $f \in \sum_{\mathcal{H}}$. Then $f \in \sum_{\mathcal{H}} \mathcal{S}^{\star}(\lambda, \alpha)$, if and only if, for $0 \leq \lambda \leq 1$ and $0 \leq \alpha < 1$,*

$$Re \left\{ \frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} \right\} \geq \alpha. \tag{3}$$

Also, we let $\sum_{\overline{\mathcal{H}}} \mathcal{S}^{\star}(\lambda, \alpha) = \sum_{\mathcal{H}} \mathcal{S}^{\star}(\lambda, \alpha) \cap \sum_{\overline{\mathcal{H}}}$.

The following theorem proved by Jahangiri and Silverman in [3] will be used throughout in this paper.

Theorem 1.1 *([3]) Let $f = h + \overline{g}$ with h and g of the form (1). If*

$$\sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1 \tag{4}$$

then f is harmonic, orientation preserving, univalent in $\widetilde{U}$ and $f \in \sum_{\mathcal{H}} \mathcal{S}^{\star}(\lambda, \alpha)$.

2 Results

We begin the results with a sufficient coefficient condition for functions in $\sum_{\mathcal{H}} \mathcal{S}^{\star}(\lambda, \alpha)$.

Theorem 2.1 *Let $f = h + \overline{g}$ be of the form (1). If*

$$\sum_{n=1}^{\infty} \frac{n + \alpha - \alpha\lambda(n+1)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n - \alpha - \alpha\lambda(n-1)}{1-\alpha} |b_n| \leq 1, \tag{5}$$

then f is harmonic, orientation preserving, univalent in $\widetilde{U}$ and $f \in \sum_{\mathcal{H}} \mathcal{S}^{\star}(\lambda, \alpha)$.

Proof. Since $n \leq \frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}$ and $n \leq \frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}$, it follows from Theorem 1.1 that f is harmonic, orientation preserving and univalent in $\widetilde{U}$. Now, we only need to show that if (5) holds then

$$Re \left\{ \frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} \right\} = Re\, \frac{A(z)}{B(z)} \geq \alpha.$$

Using the fact that $Re\ (w) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it sufficies to show that

$$|A(z)+(1-\alpha)B(z)| - |A(z)-(1+\alpha)B(z)| \geq 0, \tag{6}$$

where

$$A(z) = zh'(z) - \overline{zg'(z)}$$

and

$$B(z) = \lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)}).$$

Substituting for $A(z)$ and $B(z)$ in (6), we obtain

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)| - |A(z)-(1+\alpha)B(z)| \\
&= \Big|(2-\alpha)z - \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)+\lambda n+\lambda-1]\ a_n\ z^{-n} \\
&\quad + \sum_{n=1}^{\infty}[n-\alpha-\alpha\lambda(n-1)+\lambda n-\lambda+1]\ b_n\ \overline{z}^{\ -n}\Big| \\
&\quad - \Big| - \alpha z - \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)-\lambda n-\lambda+1]\ a_n\ z^{-n} \\
&\quad + \sum_{n=1}^{\infty}[n-\alpha-\alpha\lambda(n-1)-\lambda n+\lambda-1]\ b_n\ \overline{z}^{\ -n}\Big| \\
&\geq (2-\alpha)|z| - \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)+\lambda n+\lambda-1]\ |a_n|\ |z|^{-n} \\
&\quad - \sum_{n=1}^{\infty}[n-\alpha-\alpha\lambda(n-1)+\lambda n-\lambda+1]\ |b_n|\ |z|^{-n} - \alpha|z| \\
&\quad - \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)-\lambda n-\lambda+1]\ |a_n|\ |z|^{-n} \\
&\quad - \sum_{n=1}^{\infty}[n-\alpha-\alpha\lambda(n-1)-\lambda n+\lambda-1]\ |b_n|\ |z|^{-n} \\
&\geq 2(1-\alpha)|z|\left\{1 - \sum_{n=1}^{\infty}\frac{[n+\alpha-\alpha\lambda(n+1)]}{1-\alpha}|a_n| - \sum_{n=1}^{\infty}\frac{[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}|b_n|\right\} \\
&\geq 0, \quad \textit{by}\ (5). \quad \square
\end{aligned}$$

Next we show that the bound (5) is also necessary for functions in $\sum_{\overline{\mathcal{H}}}\mathcal{S}^{\star}(\lambda,\alpha)$.

Theorem 2.2 *Let $f = h + \overline{g}$ with h and g of the form (2). Then $f \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{n + \alpha - \alpha\lambda(n+1)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n - \alpha - \alpha\lambda(n-1)}{1-\alpha}|b_n| \leq 1. \qquad (7)$$

Proof. In view of Theorem 2.1, we only need to show that f is not in $\sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$ if condition (7) does not hold. We note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (2) to be in $\sum_{\mathcal{H}} \mathcal{S}^\star(\lambda, \alpha)$ is that the coefficient condition (3) to be satisfied. Equivalently, we must have

$$Re \left\{ \frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} - \alpha \right\}$$
$$= Re \left\{ \frac{(1-\alpha)z - \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]\ |a_n|\ z^{-n} - \sum_{n=1}^{\infty}[n-\alpha-\alpha\lambda(n-1)]\ |b_n|\ \overline{z}^{\ -n}}{z + \sum_{n=1}^{\infty}[1-\lambda n-\lambda]\ |a_n|\ z^{-n} - \sum_{n=1}^{\infty}[\lambda n-\lambda+1]\ |b_n|\ \overline{z}^{\ -n}} \right\}$$
$$\geq 0.$$

This inequality must hold for all $z \in \widetilde{U}$. Letting $z = r > 1$, we need

$$Re \left\{ \frac{1-\alpha - \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]\ |a_n|\ r^{-n-1} - \sum_{n=1}^{\infty}[n-\alpha-\alpha\lambda(n-1)]\ |b_n|\ r^{-n-1}}{1 + \sum_{n=1}^{\infty}[1-\lambda n-\lambda]\ |a_n|\ r^{-n-1} - \sum_{n=1}^{\infty}[\lambda n-\lambda+1]\ |b_n|\ r^{-n-1}} \right\} = \frac{A(r)}{B(r)} \geq 0.$$

If condition (7) does not hold then $A(r)$ is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0 > 1$ for which the quotient $\frac{A(r)}{B(r)}$ is negative. This contradicts the required condition that $\frac{A(r)}{B(r)} \geq 0$ and so the proof is complete. □

The growth result for functions in $\sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$ is discussed in the following theorem.

Theorem 2.3 *If $f \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$ then*

$$|f(z)| \leq r + (1-\alpha)r^{-1}, \quad |z| = r > 1$$

and

$$|f(z)| \geq r - (1-\alpha)r^{-1}, \quad |z| = r > 1.$$

Proof. Let $f \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$. Taking the absolute value of f we have

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=1}^{\infty} |a_n| z^{-n} - \sum_{n=1}^{\infty} |b_n| \overline{z}^{-n} \right| \\ &\leq r + \sum_{n=1}^{\infty} (|a_n| + |b_n|)\ r^{-n} \\ &\leq r + \sum_{n=1}^{\infty} (|a_n| + |b_n|)\ r^{-1} \end{aligned}$$

$$\leq \; r+\sum_{n=1}^{\infty}[(n+\alpha-\alpha\lambda(n+1))|a_n|+(n-\alpha-\alpha\lambda(n-1))|b_n|]\; r^{-1}$$
$$\leq \; r+(1-\alpha)r^{-1}$$

and

$$\begin{aligned} |f(z)| &= \left|z+\sum_{n=1}^{\infty}|a_n|z^{-n}-\sum_{n=1}^{\infty}|b_n|\bar{z}^{-n}\right| \\ &\geq r-\sum_{n=1}^{\infty}(|a_n|+|b_n|)\;\; r^{-n} \\ &\geq r-\sum_{n=1}^{\infty}(|a_n|+|b_n|)\;\; r^{-1} \\ &\geq r-\sum_{n=1}^{\infty}[(n+\alpha-\alpha\lambda(n+1))|a_n|+(n-\alpha-\alpha\lambda(n-1))|b_n|]\; r^{-1} \\ &\geq r-(1-\alpha)r^{-1}. \quad \square \end{aligned}$$

Next, we determine the extreme points of closed hulls of $\sum_{\overline{\mathcal{H}}}\mathcal{S}^{\star}(\lambda,\alpha)$ denoted by $clco\sum_{\overline{\mathcal{H}}}\mathcal{S}^{\star}(\lambda,\alpha)$.

Theorem 2.4 *$f\in clco\sum_{\overline{\mathcal{H}}}\mathcal{S}^{\star}(\lambda,\alpha)$ if and only if $f(z)=\sum_{n=0}^{\infty}(X_nh_n+Y_ng_n)$ where*

$$h_0(z)=z,\;\; h_n(z)=z+\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}\; z^{-n}\; (n=1,2,3,...),$$

$$g_0(z)=z, g_n(z)=z-\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}\; \bar{z}^{\;-n}\; (n=1,2,3,...),$$

$\sum_{n=0}^{\infty}(X_n+Y_n)=1,\;\; X_n\geq 0$ and $Y_n\geq 0$.

Proof. For h_n and g_n as given above, we may write

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(X_nh_n+Y_ng_n) \\ &= X_0h_0+Y_0g_0 \\ &\quad +\sum_{n=1}^{\infty}\left[X_n\left(z+\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}z^{-n}\right)+Y_n\left(z-\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}\bar{z}^{-n}\right)\right] \\ &= \sum_{n=0}^{\infty}(X_n+Y_n)z+\sum_{n=1}^{\infty}\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}X_nz^{-n}-\sum_{n=1}^{\infty}\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}Y_n\bar{z}^{\;-n}. \end{aligned}$$

Then

$$\sum_{n=1}^{\infty}(n+\alpha-\alpha\lambda(n+1))\left(\frac{1-\alpha}{n+\alpha-\alpha\lambda(n+1)}X_n\right)$$

$$+\sum_{n=1}^{\infty}(n-\alpha-\alpha\lambda(n-1))\left(\frac{1-\alpha}{n-\alpha-\alpha\lambda(n-1)}Y_n\right)$$

$$\begin{aligned} &= (1-\alpha)\sum_{n=1}^{\infty}(X_n+Y_n) \\ &= (1-\alpha)[1-(X_0+Y_0)] \\ &\leq 1-\alpha. \end{aligned}$$

Therefore $f \in clco \sum_{\overline{\mathcal{H}}} \mathcal{S}^{\star}(\lambda,\alpha)$.

Conversely, suppose that $f \in clco \sum_{\overline{\mathcal{H}}} \mathcal{S}^{\star}(\lambda,\alpha)$. Then we write

$$f(z) = z + \sum_{n=1}^{\infty}|a_n|z^{-n} - \sum_{n=1}^{\infty}|b_n|\bar{z}^{\,-n}$$

and

$$\sum_{n=1}^{\infty}\left[\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}|a_n| + \frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}|b_n|\right] \leq 1.$$

Setting

$$X_n = \frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}|a_n|\ , (n=1,2,3,...),$$

and

$$Y_n = \frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}|b_n|\ , (n=1,2,3,...),$$

where $0 \leq X_0 \leq 1$ and $Y_0 = 1 - X_0 - \sum_{n=1}^{\infty}(X_n+Y_n)$, we get $f(z) = \sum_{n=0}^{\infty}(X_n h_n + Y_n g_n)$ as required. □

For harmonic functions $f(z) = z + \sum_{n=1}^{\infty}|a_n|z^{-n} - \sum_{n=1}^{\infty}|b_n|\bar{z}^{\,-n}$ and $F(z) = z + \sum_{n=1}^{\infty}|A_n|z^{-n} - \sum_{n=1}^{\infty}|B_n|\bar{z}^{\,-n}$, we define the convolution of f and F as

$$(f \star F)(z) = z + \sum_{n=1}^{\infty}|a_n A_n|z^{-n} - \sum_{n=1}^{\infty}|b_n B_n|\bar{z}^{\,-n}. \tag{8}$$

In the next theorem, we examine the convolution properties of the class $\sum_{\overline{\mathcal{H}}} \mathcal{S}^{\star}(\lambda,\alpha)$.

Theorem 2.5 *For $0 \leq \beta \leq \alpha < 1$, let $f \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^{\star}(\lambda,\alpha)$ and $F \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^{\star}(\lambda,\alpha)$. Then $(f \star F) \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^{\star}(\lambda,\alpha) \subset \sum_{\overline{\mathcal{H}}} \mathcal{S}^{\star}(\lambda,\alpha)$.*

Proof. Write $f(z) = z + \sum_{n=1}^{\infty}|a_n|z^{-n} - \sum_{n=1}^{\infty}|b_n|\bar{z}^{\,-n}$ and $F(z) = z + \sum_{n=1}^{\infty}|A_n|z^{-n} - \sum_{n=1}^{\infty}|B_n|\bar{z}^{\,-n}$. Then the convolution of f and F is given by (8).

Note that $|A_n| \leq 1$ and $|B_n| \leq 1$ since $F \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$. Then we have

$$\sum_{n=1}^{\infty} [[n + \alpha - \alpha\lambda(n+1)]|a_n||A_n| + [n - \alpha - \alpha\lambda(n-1)]|b_n||B_n|]$$

$$\leq \sum_{n=1}^{\infty} [[n + \alpha - \alpha\lambda(n+1)]|a_n| + [n - \alpha - \alpha\lambda(n-1)]|b_n|] .$$

Therefore, $(f \star F) \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha) \subset \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$ since the right hand side of the above inequality is bounded by $1 - \alpha$ while $1 - \alpha \leq 1 - \beta$. □

Now, we determine the convex combination properties of the members of $\sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$.

Theorem 2.6 *The class* $\sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$ *is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$, suppose that $f_i \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$ where f_i is given by

$$f_i(z) = z + \sum_{n=1}^{\infty} |a_{n,i}| z^{-n} - \sum_{n=1}^{\infty} |b_{n,i}| \bar{z}^{\ -n}.$$

For $\sum_{i=1}^{\infty} c_i = 1,\ 0 \leq c_i \leq 1$, the convex combinations of f_i may be written as

$$\begin{aligned}
\sum_{i=1}^{\infty} c_i f_i(z) &= c_1 z + \sum_{n=1}^{\infty} c_1 |a_{n,1}| z^{-n} - \sum_{n=1}^{\infty} c_1 |b_{n,1}| \bar{z}^{\ -n} + c_2 z + \sum_{n=1}^{\infty} c_2 |a_{n,2}| z^{-n} - \sum_{n=1}^{\infty} c_2 |b_{n,2}| \bar{z}^{\ -n} ... \\
&= z \sum_{i=1}^{\infty} c_i + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{n,i}| \right) z^{-n} - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{n,i}| \right) \bar{z}^{\ -n} \\
&= z + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{n,i}| \right) z^{-n} - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{n,i}| \right) \bar{z}^{\ -n}.
\end{aligned}$$

Next, consider

$$\sum_{n=1}^{\infty} \left([n + \alpha - \alpha\lambda(n+1)] \left| \sum_{i=1}^{\infty} c_i |a_{n,i}| \right| \right) + \sum_{n=1}^{\infty} \left([n - \alpha - \alpha\lambda(n-1)] \left| \sum_{i=1}^{\infty} c_i |b_{n,i}| \right| \right)$$

$$\begin{aligned}
&= c_1 \sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] |a_{n,1}| + ... + c_m \sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] |a_{n,m}| + ... \\
&\quad + c_1 \sum_{n=1}^{\infty} [n - \alpha - \alpha\lambda(n-1)] |b_{n,1}| + ... + c_m \sum_{n=1}^{\infty} [n - \alpha - \alpha\lambda(n-1)] |b_{n,m}| + ... \\
&= \sum_{i=1}^{\infty} c_i \left\{ \sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] |a_{n,i}| + \sum_{n=1}^{\infty} [n - \alpha - \alpha\lambda(n-1)] |b_{n,i}| \right\}.
\end{aligned}$$

Now, $f_i \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$, therefore from Theorem 2.2, we have

$$\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]|a_{n,i}| + \sum_{n=1}^{\infty}[n-\alpha-\alpha\lambda(n-1)]|b_{n,i}| \leq 1-\alpha \ .$$

Hence

$$\sum_{n=1}^{\infty}\left([n+\alpha-\alpha\lambda(n+1)]\left|\sum_{i=1}^{\infty} c_i|a_{n,i}|\right|\right) + \sum_{n=1}^{\infty}\left([n-\alpha-\alpha\lambda(n-1)]\left|\sum_{i=1}^{\infty} c_i|b_{n,i}|\right|\right)$$

$$\leq (1-\alpha)\sum_{i=1}^{\infty} c_i$$

$$= 1-\alpha.$$

By using Theorem 2.2 again, we have $\sum_{i=1}^{\infty} c_i f_i \in \sum_{\overline{\mathcal{H}}} \mathcal{S}^\star(\lambda, \alpha)$. □

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