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General System for Strongly Accretive Nonlinear Variational Inequalities in $q-$Uniformly Smooth Banach Spaces[1]

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Abstract. The approximate solvability of a generalized system for strongly accretive nonlinear variational inequalities in $q-$uniformly smooth Banach spaces is studied, based on the convergence of sunny nonexpansive retraction projection methods. The results presented in this paper extend and improve the main results of R.U.Verma[General convergence analysis for two-step projection methods and applications to variational problems, Appl.Math.Lett. 18(2005):1286-1292;Projection Methods, Algorithms, and a new system of nonlinear variational inequalities, J.Comp.Math.Appl.41(2004):1025-1031].

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1. Introduction

Projection methods have played a significant role in the numerical resolution of variational inequalities in Hilbert spaces. And Verma[1] introduces the general two-step model for projection methods, which reduces to the two-step model applied in [2] and then applies it to the approximation solvability of a two-step strongly monotonic nonlinear variational inequality in a Hilbert space setting.

It is the aim of this paper to improve the results of Verma[1,2] in $q-$uniformly smooth Banach spaces. In order to overcome the difficulties caused by the lack

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of projections, we will restrict our investigation in smooth Banach space because in such a space, the fixed point set of a nonexpansive mapping is a sunny nonexpansive retract(see definition in Section 2). Since a sunny nonexpansive retraction in terms of a duality mapping enjoys some of the nice properties that a projection in a Hilbert space has, we are able to establish the main results in a smooth Banach space setting.

Let X be a real smooth Banach space with dual X^*, we denote by J the normalized duality mapping from X to 2^{X^*}. It is well known that if X is smooth, the J is single-valued. In the sequel, we shall denote the single-valued normalized duality by j. Let K be a nonempty closed convex subset of X and Let $A: K \to K$ be any mapping on K. We consider a system of two nonlinear variational inequality (abbreviated as SNVI) problems as follows: to find elements $x^*, y^* \in K$ such that

$$\langle \rho A(y^*) + x^* - y^*, j(x - x^*)\rangle \geq 0,\ \forall\ x \in K\ and\ for\ \rho > 0 \tag{1.1}$$

$$\langle \eta A(x^*) + y^* - x^*, j(x - y^*)\rangle \geq 0,\ \forall\ x \in K\ and\ for\ \eta > 0 \tag{1.2}$$

The SNVI problem (1.1) and (1.2) is equivalent to the following sunny nonexpansive retraction projection formulas

$$x^* = P_K[y^* - \rho A(y^*)]\ for\ \rho > 0$$

$$y^* = P_K[x^* - \eta A(x^*)]\ for\ \eta > 0$$

where P_K is the sunny nonexpansive retraction projection from X onto K.

Next we consider two special cases of SNVI problem (1.1) and (1.2)

(1) If $\eta = 0$, then the SNVI problem (1.1) and (1.2) reduces to the following nonlinear variational inequality (NVI) problem: to find an $x^* \in K$ such that

$$\langle A(x^*), j(x - x^*)\rangle \geq 0,\ \forall x \in K \tag{1.3}$$

(2) If K is a closed convex cone of X, then the SNVI problem (1.1) and (1.2) is equivalent to the following system of nonlinear complementarity (SNC) problems: to find $x^*, y^* \in K$ such that $A(x^*) \in K^*, A(y^*) \in K^*$ and

$$\langle \rho A(y^*) + x^* - y^*, j(x^*)\rangle = 0,\ for\ \rho > 0 \tag{1.4}$$

$$\langle \eta A(x^*) + y^* - x^*, j(y^*)\rangle = 0,\ for\ \eta > 0 \tag{1.5}$$

where K^* is the polar cone to K defined by

$$K^* = \{f \in X : \langle f, j(x)\rangle \geq 0,\ \forall\ x \in K\}$$

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2. Preliminaries

Throughout this paper, we always let X be a real Banach space with the dual space X^*. The generalized duality mappings $J_q(x): X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f \in X^*, \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\},\ \forall x \in X$$

where $q > 1$ is a constant. In particular, $J_2 = J$ is the usual normalized duality mapping. It is known that $J_q = \|x\|^{q-2} J$ for all $x \in X$, and $J_q(x)$ is single-valued if X^* is strictly convex. In the sequel, unless otherwise specified, we always suppose that X is a real Banach space such that J_q is single-valued. We denote the single-valued generalized duality by j_q. If X is a Hilbert space, then J becomes the identity mapping of X.

The modulus of smoothness of X is the function $\rho_X : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_X(t) = sup\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1 : \|x\| = 1, \|y\| = t, x \in X, y \in X\}, \ t > 0.$$

If there exists constant $c > 0$ and a real number $1 < q < \infty$, such that $\rho_X(t) \le ct^q$, then X is said to be $q-$uniformly smooth. A Banach space X is called uniformly smooth if $lim_{t\to 0}\rho_X(t)/t = 0$

In the sequel, we will give some definitions.

Definition 2.1. Let $A : X \to X$ be a single-valued operation, then the operator A is said to be
(1) accretive if

$$\langle Ax - Ay, J_q(x-y)\rangle \ge 0, \ \forall x, y \in X;$$

(2) strictly accretive if

$$\langle Ax - Ay, J_q(x-y)\rangle \ge 0, \ \forall x, y \in X,$$

and the equality holds if and only if $y = x$;
(3) strongly accretive if there exists a constant $r > 0$, such that

$$\langle Ax - Ay, J_q(x-y)\rangle \ge r\|x-y\|^q, \ \forall x, y \in X;$$

(4) Lipschitz continuous if there exists a constant $s > 0$, such that

$$\|Ax - Ay\| \le s\|x-y\|, \ \forall x, y \in X;$$

Definition 2.2. Let C and K be nonempty subsets of a Banach space X such that C is nonempty closed convex and $K \subset C$, then a mapping $P_K : C \to K$ is called
(1) retraction from C onto K if

$$P_K x = x, \ \forall x \in K.$$

(2) sunny if

$$P_K(P_K x + t(x - P_K x)) = P_K x, \ \forall x \in C$$

whenever $P_K x + t(x - P_K x) \in C$ and $t > 0$.
(3) a sunny nonexpansive retraction if P_K is sunny, nonexpansive and a retraction of C onto K.

The following lemma is well known (see reference [3,4]).

Lemma 2.1. *Let C be a nonempty convex subset of a smooth Banach space X, $K \subset C$, $J : X \to X^*$ the (normalized) duality mapping of X, and $P_K : C \to K$ a retraction. Then the following are equivalent:*

(1)$\langle x - P_K x, j(y - P_K x)\rangle \leq 0$ *for all* $x \in C$ *and* $y \in K$;
(2)P_K *is both sunny and nonexpansive.*

In order to prove our main results, we need the following lemmas.

Lemma 2.2. *([5]) Let $\{\lambda_n\}$ be a sequence in [0,1) such that $lim_{n\to\infty}\lambda_n = 0$, then*

$$\sum_{n=1}^{\infty} \lambda_n = \infty \Leftrightarrow \prod_{n=1}^{\infty}(1 - \lambda_n) = 0.$$

Lemma 2.3. *([6]) Let X be a real uniformly smooth Banach space. Then, X is $q-$uniformly smooth if and only if there exists a constant $c > 0$, such that for all $x, y \in X$*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c\,\|y\|^q\,.$$

3. Algorithms

In this section, we deal with an introduction of general two-step models for sunny nonexpansive retraction projection and its special forms that can be applied to the convergence analysis for sunny nonexpansive retraction projection in the context of the approximation solvability of the SNVI problem (1.1) and (1.2).

Algorithm 3.1 For arbitrarily chosen initial points $x_1, y_1 \in K$, computing the sequences $\{x_n\}, \{y_n\}$ such that

$$x_{n+1} = (1 - a_n)x_n + a_n P_K[y_n - \rho A(y_n)]$$
$$y_n = (1 - b_n)x_n + b_n P_K[x_n - \eta A(x_n)]$$

where P_K is the sunny nonexpansive retraction mapping of X onto K, ρ and $\eta > 0$ are constants and $\{a_n\}, \{b_n\}$ are sequences in [0,1].

For $\{b_n\} = 1$ in Algorithm 3.1, we get

Algorithm 3.2 For arbitrarily chosen initial points $x_1, y_1 \in K$, computing the sequences $\{x_n\}, \{y_n\}$ such that

$$x_{n+1} = (1 - a_n)x_n + a_n P_K[y_n - \rho A(y_n)]$$
$$y_n = P_K[x_n - \eta A(x_n)]$$

where P_K is the sunny nonexpansive retraction mapping of X onto K, ρ and $\eta > 0$ are constants and $\{a_n\}$ is a sequence in [0,1].

For $\eta = 0$ and $\{b_n\} = 1$ in Algorithm 3.1, we get

Algorithm 3.3 For an arbitrarily chosen initial point $x_1 \in K$, computing the sequence $\{x_n\}$ such that

$$x_{n+1} = (1 - a_n)x_n + a_n P_K[x_n - \rho A(x_n)]$$

where P_K is the sunny nonexpansive retraction mapping of X onto K, $\rho > 0$ is a constant and $\{a_n\}$ is a sequence in [0,1].

4. Main Results

We now present, based on Algorithm 3.1, the approximation-solvability of the SNVI problem (1.1) and (1.2) involving a mapping $A: K \to X$ which is strongly accretive and Lipschitz continuous with constants r and s, respectively, in a $q-$ uniformly smooth Banach space setting.

Theorem 4.1. *Let X be a real $q-$uniformly smooth Banach, and K be a nonempty closed convex subset of X and $T: K \to X$ be a strongly accretive with a constant r and $s-$Lipschitz continuous mapping. $P_K: X \to K$ sunny nonexpansive retraction mapping. Suppose that $x^*, y^* \in K$ form a solution to the SNVI problem (1.1) and (1.2), the sequences $\{x_n\}, \{y_n\}$ are generated by Algorithm 3.1 and $a_n, b_n \in [0,1]$, $\sum_{n=0}^{\infty} a_n b_n = \infty$. Then, sequences $\{x_n\}, \{y_n\}$, respectively, converge to x^* and y^* for*

$$0 < \rho^{q-1} < \frac{rq}{cs^q},\ 0 < \eta^{q-1} < \frac{rq}{cs^q}.$$

Proof. Since x^* and y^* form a solution to the SNVI problem (1.1) and (1.2), it follows that

$$\begin{array}{l} x^* = P_K[y^* - \rho A(y^*)] \\ y^* = P_K[x^* - \rho A(x^*)] \end{array}$$

Applying Algorithm 3.1, we have

$$\begin{array}{ll} & \|x_{n+1} - x^*\| \\ = & \|(1-a_n)x_n + a_n P_K[y_n - \rho A(y_n)] - (1-a_n)x^* - a_n P_K[y^* - \rho A(y^*)]\| \\ \leq & (1-a_n)\|x_n - x^*\| + a_n \|P_K[y_n - \rho A(y_n)] - P_K[y^* - \rho A(y^*)]\| \\ \leq & (1-a_n)\|x_n - x^*\| + a_n \|y_n - y^* - \rho[A(y_n) - A(y^*)]\| \end{array} \tag{4.1}$$

Since A is strongly accretive and $s-$Lipschitz continuous, we have

$$\begin{array}{ll} & \|y_n - y^* - \rho[A(y_n) - A(y^*)]\|^q \\ \leq & \|y_n - y^*\|^q - q\rho\langle A(y_n) - A(y^*), j_q(y_n - y^*)\rangle + c\rho^q \|A(y_n) - A(y^*)\|^q \\ \leq & \|y_n - y^*\|^q - qr\rho \|y_n - y^*\|^q + cs^q\rho^q \|y_n - y^*\|^q \\ = & (1 - rq\rho + c\rho^q s^q)\|y_n - y^*\|^q \end{array}$$

Thus,

$$\|y_n - y^* - \rho[A(y_n) - A(y^*)]\| \leq (1 - rq\rho + c\rho^q s^q)^{\frac{1}{q}} \|y_n - y^*\| \tag{4.2}$$

Substituting (4.2) into (4.1) and simplifying the result, we have

$$\begin{array}{lll} \|x_{n+1} - x^*\| & \leq & (1-a_n)\|x_n - x^*\| + a_n\theta \|y_n - y^*\| \\ & \leq & (1-a_n)\|x_n - x^*\| + a_n \|y_n - y^*\| \end{array} \tag{4.3}$$

where $0<\theta=(1-rq\rho+c\rho^q s^q)^{\frac{1}{q}}<1$.
Similarly, we have

$$\begin{aligned}\|y_n-y^*\| &= \|(1-b_n)(x_n-x^*)+b_n(P_K[x_n-\eta A(x_n)]-P_K[x^*-\rho A(x^*)])\| \\ &\leq (1-b_n)\|x_n-x^*\|+b_n\|x_n-x^*-\eta[A(x_n)-A(x^*)]\| \\ &\leq (1-b_n)\|x_n-x^*\|+b_n(1-rq\eta+c\rho^q s^q)^{\frac{1}{q}}\|x_n-x^*\| \\ &= (1-b_n)\|x_n-x^*\|+b_n\sigma\|x_n-x^*\| \end{aligned} \tag{4.4}$$

where $0<\sigma=(1-rq\eta+c\rho^q s^q)^{\frac{1}{q}}<1$.
It follows from (4.3) and (4.4) that

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq (1-a_n)\|x_n-x^*\|+a_n\|y_n-y^*\| \\ &\leq (1-a_n)\|x_n-x^*\|+a_n(1-b_n+b_n\sigma)\|x_n-x^*\| \\ &= [1-(1-\sigma)a_nb_n)\|x_n-x^*\| \\ &\leq \textstyle\prod_{i=1}^n[1-(1-\sigma)a_ib_i]\|x_1-x^*\|\end{aligned} \tag{4.5}$$

where $0<\sigma=(1-rq\eta+c\rho^q s^q)^{\frac{1}{q}}<1$.
Since $0<\sigma<1$ and $\sum_{n=1}^{\infty}a_nb_n=\infty$, thus by Lemma 2.3,

$$\prod_{i=1}^{\infty}[1-(1-\sigma)a_ib_i]=0$$

Hence, the sequence $\{x_n\}$ converges to x^* by (4.5), and the sequence $\{y_n\}$ converges to y^* by (4.4). □

Remark 4.1. Theorem 4.1 extends and improves the main results in Verma[1] from Hilbert space to $q-$uniformly smooth Banach space.

The following theorems can be obtained from Theorem 4.1 immediately.

Theorem 4.2. *Let X be a real $q-$uniformly smooth Banach, and K be a nonempty closed convex subset of X and $T: K\to X$ be a strongly accretive with a constant r and $s-$Lipschitz continuous mapping. $P_K: X\to K$ sunny nonexpansive retraction mapping. Suppose that $x^*,y^*\in K$ form a solution to the SNVI problem (1.1) and (1.2), the sequences $\{x_n\},\{y_n\}$ are generated by Algorithm 3.2 and $a_n,b_n\in[0,1]$, $\sum_{n=0}^{\infty}a_nb_n=\infty$. Then, sequences $\{x_n\},\{y_n\}$, respectively, converge to x^* and y^* for*

$$0<\rho^{q-1}<\frac{rq}{cs^q},\ 0<\eta^{q-1}<\frac{rq}{cs^q}.$$

Remark 4.2. Theorem 4.2 extends and improves Theorem 2.1 in Verma[2] from Hilbert space to $q-$uniformly smooth Banach space.

Theorem 4.3. *Let X be a real $q-$uniformly smooth Banach, and K be a nonempty closed convex subset of X and $T: K\to X$ be a strongly accretive with a constant r and $s-$Lipschitz continuous mapping. $P_K: X\to K$ sunny nonexpansive retraction mapping. Suppose that $x^*,y^*\in K$ form a solution to* *the SNVI problem (1.3), the sequences* $\{x_n\}$ *are generated by Algorithm 3.3 and* $a_n \in [0, 1]$, $\sum_{n=0}^{\infty} a_n = \infty$. *Then, sequences* $\{x_n\}$ *converge to* x^* *for*

$$0 < \rho^{q-1} < \frac{rq}{cs^q}.$$

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