On Banach-Steinhaus Theorems in Generalized 2-Normed Spaces

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Abstract. In this study, we will review Banach-Steinhaus theorems of bounded linear operators from a normed space into a generalized 2-normed space. This study is a review on the three works of Zofia Lewandowska.

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1. INTRODUCTION

Definition 1. ([1] and [2]) Let X and Y be linear spaces, D be non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets

$$D_x = \{y \in Y : (x, y) \in D\}, \quad D^y = \{x \in X : (x, y) \in D\}$$

are linear subspaces of the spaces Y and X , respectively. A function $\|\cdot, \cdot\| : D \rightarrow [0, \infty)$ is called a generalized 2-norm on D if it satisfies the following conditions:

- (N1) $\|x, \alpha y\| = |\alpha| \cdot \|x, y\| = \|\alpha x, y\|$ for all $(x, y) \in D$ and every scalar α ,
- (N2) $\|x, y + z\| = \|x, y\| + \|x, z\|$ for all $(x, y), (x, z) \in D$,
- (N3) $\|x + y, z\| = \|x, z\| + \|y, z\|$ for all $(x, z), (y, z) \in D$,

Then $(D, \|\cdot, \cdot\|)$ is called a 2-normed set. In particular, if $D = X \times Y$, then $(X \times Y, \|\cdot, \cdot\|)$ is called a generalized 2-normed space. Moreover if $X = Y$, then the generalized 2-normed space is denoted by $(X, \|\cdot, \cdot\|)$.

Theorem 1. ([1] and [2]) Let $(X \times Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space.

(a) A family β of all sets defined by $\bigcap_{i=1}^n \{x \in X : \|x, y_i\| < \varepsilon\}$, where $n \in \mathbb{N}$, $y_1, \dots, y_n \in Y$ and $\varepsilon > 0$, forms a complete system of neighborhoods of zero for a locally convex topology in Y .

(b) A family β of all sets defined by $\bigcap_{i=1}^n \{y \in Y : \|x_i, y\| < \varepsilon\}$, where $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$, forms a complete system of neighborhoods of zero for a locally convex topology in X .

We will denote the above topologies by the symbols $\tau(X, Y)$ and $\tau(Y, X)$, respectively. In the case when $X = Y$, we will denote these topologies by $\tau_1(X) = \tau(X, Y)$ and $\tau_2(X) = \tau(Y, X)$.

Definition 2. (i) Let $(XxY, \|\cdot, \cdot\|)$ be a generalized 2-normed space and Σ a directed set. A net $\{x_\sigma\}_{\sigma \in \Sigma}$ is convergent to $x_0 \in X$ in $(X, \tau(X, Y))$ if and only if for all $y \in Y$ and each $\varepsilon > 0$ there exists $\sigma_0 \in \Sigma$ such that $\|x_\sigma - x_0, y\| < \varepsilon$ for all $\sigma \geq \sigma_0$. Also, a net $\{y_\sigma\}_{\sigma \in \Sigma}$ is convergent to $y_0 \in Y$ in $(Y, \tau(Y, X))$ if and only if for all $x \in X$ and $\varepsilon > 0$ there exists $\sigma_0 \in \Sigma$ such that $\|x, y_\sigma - y_0\| < \varepsilon$ for all $\sigma \geq \sigma_0$ ([2]).

(ii) Let $(XxY, \|\cdot, \cdot\|)$ be generalized 2-normed space. A sequence $\{x_n\}_{n \geq 1}$ in X is called a Cauchy sequence if for every $y \in Y$ and $\varepsilon > 0$ there exists a $n \in \mathbb{N}$ such that $\|x_n - x_m, y\| < \varepsilon$ for all $n, m \geq n_0$.

(iii) Let $(XxY, \|\cdot, \cdot\|)$ be generalized 2-normed space. The space $(X, \tau(X, Y))$ is called sequentially complete if every Cauchy sequence in X is convergent in this space.

(iv) Let X be a real normed space and $\Gamma \subseteq YxY$ be a 2-normed set, where Y denotes a real linear space. Define

$$F = \{(f, g) \in L(X, Y)^2 : \forall x \in X \quad (f(x), g(x)) \in \Gamma$$

$$\& \exists M > 0, \forall x \in X \quad \|f(x), g(x)\| \leq M \cdot \|x\|^2\}$$

Then, $(F, \|\cdot, \cdot\|)$ is a 2-normed set, where the function $\|\cdot, \cdot\| : F \longrightarrow [0, \infty)$ is defined by

$$\|f, g\| = \inf \{M > 0 : \forall x \in X \quad \|f(x), g(x)\| \leq M \cdot \|x\|^2\}.$$

Note that the sets

$$F^g = \{f' \in L(X, Y) : (f', g) \in F\} \quad \& \quad F_f = \{g' \in L(X, Y) : (f, g') \in F\}$$

are linear subspaces of the space $L(X, Y)$, for every $(f, g) \in L(X, Y)^2$.

Similarly we can define

$$\Theta = \{(f, g) \in L(X, Y)^2 : \forall x \in X \quad (f(x), g(x)) \in \Gamma$$

$$\& \exists M > 0, \forall x \in X \quad \|f(x), g(y)\| \leq M \cdot \|x\| \cdot \|y\|\}$$

Then, $(\Theta, \|\cdot, \cdot\|)$ is a 2-normed set, where the function $\|\cdot, \cdot\| : \Theta \longrightarrow [0, \infty)$ is defined by

$$\|f, g\| = \inf \{M > 0 : \forall x \in X \quad \|f(x), g(y)\| \leq M \cdot \|x\| \cdot \|y\|\}.$$

Similarly the sets

$$\Theta^g = \{f' \in L(X, Y) : (f', g) \in \Theta\} \quad \& \quad \Theta_f = \{g' \in L(X, Y) : (f, g') \in \Theta\}$$

are linear subspaces of the space $L(X, Y)$, for every $(f, g) \in L(X, Y)^2$.

2. SOME RESULTS

Theorem 2. Let $(X, \|\cdot\|)$ be a Banach space, $(Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space and $\{f_n\}_{n \geq 1}$ a sequence of elements from Θ^g for some $g \in L(X, Y)$. Then the following conditions are equivalent:

- (a) The sequence of 2-norms $\{\|f_n, g\|\}_{n \geq 1}$ is bounded.
- (b) $\exists M > 0 \forall x, y \in X, \|x\| \leq 1, \|y\| \leq 1, \forall n \geq 1, \|f_n(x), g(y)\| \leq M$.
- (c) The following conditions are true:
 - (i) $\forall x \in X, \exists M_x > 0, \forall y \in X, \|y\| \leq 1, \forall n \geq 1, \|f_n(x), g(y)\| \leq M_x$.
 - (ii) $\forall y \in X, \exists M_y > 0, \forall x \in X, \|x\| \leq 1, \forall n \geq 1, \|f_n(x), g(y)\| \leq M_y$.

Proof. At first, let us suppose that the sequence of 2-norms $\{\|f_n, g\|\}_{n \geq 1}$ is bounded. From this, it follows that there exists a positive number M such that $\|f_n, g\| \leq M$ for each $n \in N$. Thus for $x, y \in X, \|x\| \leq 1, \|y\| \leq 1$ and $n \in N$, we obtain $\|f_n(x), g(y)\| \leq \|f_n, g\| \cdot \|x\| \cdot \|y\| \leq M$. Now, let the condition (b) be satisfied. We fix $x \in X \setminus \{0\}$. Then for each $y \in X, \|y\| \leq 1$ and $n \in N$ we obtain the inequalities:

$$\|f_n(x), g(y)\| = \left\| f_n \left(\frac{x}{\|x\|} \cdot \|x\| \right), g(y) \right\| = \|x\| \cdot \left\| f_n \left(\frac{x}{\|x\|}, g(y) \right) \right\| \leq M \cdot \|x\|$$

If we choose $M_x = M \cdot \|x\|$, then we have the condition (i). Moreover, for $x = 0$ the condition (i) is satisfied for every positive number M_x . In similar way, taking $M_y = M \cdot \|y\|$ for each $y \in X \setminus \{0\}$ and any positive number for $y = 0$, we obtain (ii). Conversely, let (i) and (ii) be satisfied. In XX , let us define a norm by the formula,

$$\|(x, y)\|_* = \|x\| + \|y\| \text{ for each } (x, y) \in XX.$$

It is easy to verify that $(XX, \|\cdot\|_*)$ is a Banach space. Put

$$A_{nm} = \{(x, y) \in XX : \|f_n(x), g(y)\| \leq m\} \text{ and } B_m = \bigcap_{n=1}^{\infty} A_{nm}$$

for $m, n \in N$. We shall show that the sets B_m are closed in $(XX, \|\cdot\|_*)$ for each $m \in N$. At first, we shall show that sets A_{nm} are closed in this space. Let $m, n \in N$ and let $\{(x_k, y_k) : k \in N\} \subset A_{nm}$ be a sequence converging to $(x', y') \in XX$. Then

$$\|f_n(x_k), g(y_k)\| \leq m \text{ and } \|(x_k, y_k) - (x', y')\|_* \rightarrow 0, k \rightarrow \infty.$$

The last condition is equivalent to the following: $\|x_k - x'\| \rightarrow 0$ and $\|y_k - y'\| \rightarrow 0$, which implies the convergence of the sequence $\{x_k : k \in N\}, \{y_k : k \in N\}$.

As a consequence these sequences are bounded. There exists $K > 0$ such that the inequalities $\|x_k\| \leq K$, $\|y_k\| \leq K$ are true for each $k \in N$. Using these results we get

$$\begin{aligned} \|f_n(x'), g(y')\| &\leq M + K \cdot \|f_n, g\| \cdot \|x_k - x'\| + K \cdot \|f_n, g\| \cdot \|y_k - y'\| \\ &\quad + \|f_n, g\| \cdot \|x_k - x'\| \cdot \|y_k - y'\|. \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain $\|f_n(x'), g(y')\| \leq M$, which means that $(x', y') \in A_{nm}$. Therefore the sets A_{nm} are closed for each $n, m \in N$, and hence the sets B_m are closed in $(X \times X, \|\cdot\|_*)$. ■

Now we shall show that the equality $X \times X = \bigcup_{m=1}^{\infty} B_m$ is true. Let

$x, y \in X$, $x \neq 0$. Then $\left\| \frac{x}{\|x\|} \right\| = 1$. By virtue (ii) there exists $M_y > 0$ such that $\left\| f_n \left(\frac{x}{\|x\|} \right), g_y \right\| \leq M_y$ for each $n \in N$. Thus $\|f_n(x), g(y)\| \leq M_y \cdot \|x\|$ for each $n \in N$. If $x = 0$, then $\|x\| \leq 1$ and $\|f_n(x), g(y)\| = \|0, g(y)\| = 0 = M_y \cdot \|0\|$. as a consequence, for every $x, y \in X$ the sequence $\{\|f_n(x), g(y)\| : n \in N\}$ is bounded. From this it follows that for any point $(x, y) \in X \times X$ there exists $m \in N$ such that $\|f_n(x), g(y)\| \leq M$ for every $m \in n$, i.e. $(x, y) \in \bigcup_{m=1}^{\infty} B_m$.

Thus $X \times X = \bigcup_{m=1}^{\infty} B_m$. By the well known Baire theorem there exists a set B_{m_0} with non-empty interior. Therefore B_{m_0} contains some closed ball with the center (x_0, y_0) and radius r . Denote it by $K((x_0, y_0), r)$. Thus for each $n \in N$ and $(x, y) \in K((x_0, y_0), r)$, we have $\|f_n(x), g(y)\| \leq m_0$. Let us take $x, y \in X$ such that $\|x\| \leq \frac{r}{2}$ and $\|y\| \leq \frac{r}{2}$. Then

$$\|(x, y)\|_* = \|x\| + \|y\| \leq r \quad \text{and} \quad \|(x, y)\|_* = \|(x + x_0, y + y_0) - (x_0, y_0)\|_* \leq r.$$

Therefore $\|f_n(x + x_0), g(y + y_0)\| \leq m_0$. In particular, $\|f_n(x_0), g(y_0)\| \leq m_0$. Thus,

$$\begin{aligned} \|f_n(x), g(y)\| &\leq \|f_n(x + x_0), g(y + y_0)\| + \|f_n(x + x_0), g(y_0)\| \\ &\quad + \|f_n(x_0), g(y + y_0)\| + \|f_n(x_0), g(y_0)\| \\ &\leq 2m_0 + \|f_n(x_0) + f_n(x_0), g(y_0)\| + \|f_n(x_0), g(y) + g(y_0)\| \\ &\leq 4m_0 + \|f_n(x), g(y_0)\| + \|f_n(x_0), g(y)\|. \end{aligned}$$

So we have shown that the inequalities $\|x\| \leq \frac{r}{2}$ and $\|y\| \leq \frac{r}{2}$ imply the condition

$$\|f_n(x), g(y)\| \leq 4m_0 + \|f_n(x), g(y_0)\| + \|f_n(x_0), g(y)\|.$$

Now, let $x, y \in X$, $\|x\| \leq 1$ and $\|y\| \leq 1$. Because $\left\| \frac{r}{2}x \right\| \leq \frac{r}{2}$ and $\left\| \frac{r}{2}y \right\| \leq \frac{r}{2}$, then

$$\left\| f_n \left(\frac{r}{2}x \right), g \left(\frac{r}{2}y \right) \right\| \leq 4m_0 + \left\| f_n \left(\frac{r}{2}x \right), g(y_0) \right\| + \left\| f_n(x_0), g \left(\frac{r}{2}y \right) \right\|.$$

As a sequence, we obtain

$$\|f_n(x), g(y)\| \leq \frac{16m_0}{r^2} + \frac{2}{r} (\|f_n(x), g(y_0)\| + \|f_n(x), g(y)\|) \quad \text{for each } n \in N.$$

Applying (i) we have that there exists $M_{x_0} > 0$ such that for every $y \in X$, $\|y\| \leq 1$ and $n \in N$ the inequality $\|f_n(x_0), g(y)\| \leq M_{x_0}$ is true. However the assumption (ii) implies there exists $M_{y_0} > 0$ such that for every $x \in X$, $\|x\| \leq 1$ and $n \in N$ the inequality $\|f_n(x), g(y_0)\| \leq M_{y_0}$ is satisfied. So

$$\|f_n(x), g(y)\| \leq \frac{16m_0}{r^2} + \frac{2}{r} (M_{y_0} + M_{x_0})$$

for each $n \in N$ and $x, y \in X$ such that $\|x\| \leq 1$, $\|y\| \leq 1$. Therefore

$$\begin{aligned} \|f_n, g\| &= \sup \{ \|f_n(x), g(y)\| : x, y \in X, \|x\| \leq 1, \|y\| \leq 1 \} \\ &\leq \frac{16m_0 + 2r(M_{x_0} + M_{y_0})}{r^2} \end{aligned}$$

for each $n \in N$. So the sequence $\{\|f_n, g\| : n \in N\}$ is bounded and the proof is complete.

Theorem 3. ([3]) *Let $(X, \|\cdot\|)$ be a Banach space, $(Y, \|\cdot, \cdot\|)$ a generalized 2-normed space and g a linear operator from X to Y . If $\{f_n : n \in N\} \subset \theta^g$ is pointwise convergent to $f \in L(X, Y)$ and satisfies one of the conditions (a), (b), (c) from Theorem 2.1 then $f \in \theta^g$.*

Theorem 4. *Let A be a linearly dense set in a Banach space $(X, \|\cdot\|)$, $(Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space such that $(Y, \tau_1(Y))$ is a Hausdorff sequentially complete space. Let g be a linear operator from X into Y and $\{f_n\}_{n \geq 1} \subseteq \Theta^g$. Then the following conditions are equivalent:*

(a) *The sequence $\{f_n\}_{n \geq 1}$ is pointwise convergent to $f \in L(X, Y)$ and the conditions (i), (ii) from Theorem 2 are satisfied.*

(b) *The sequence $\{f_n\}_{n \geq 1}$ is pointwise convergent to $f \in \Theta^g$ on the set A and the sequence of 2-norms $\{\|f_n, g\|\}_{n \geq 1}$ is bounded.*

Proof. If the sequence $\{f_n(x) : n \in N\}$ is convergent to $f(x) \in Y$ for each $x \in X$, then it convergent also for $x \in A \subset X$. Moreover this follows from theorem 2.1 and theorem 3-the sequence $\{\|f_n, g\| : n \in N\}$ is bounded and $f \in \theta^g$. Now, we will suppose that the sequence $\{f_n : n \in N\}$ is pointwise convergent to $f \in \theta^g$ on the set A and the conditions (i), (ii) hold. Let X_0 be the vector subspace of the Banach space X generated by A . So X_0 is normed space. Let $x, y \in X_0$. Then $x = a_1x_1 + \dots + a_kx_k$, $y = b_1y_1 + \dots + b_t y_t$, where $a_i, b_j \in R$, $x_i, y_j \in A$, $i = 1, 2, 3, \dots, k$; $j = 1, 2, 3, \dots, t$ and $k, t \in N$. Thus, it follows from assumptions on f_n, f, g that $\|f_n(x) - f(x), g(y)\| = \|a_1(f_n(x_1) - f(x_1)) + \dots + a_k(f_n(x_k) - f(x_k)), b_1g(y_1) + b_2g(y_2) + \dots + b_tg(y_t)\|$.

Using properties of 2-norms, we get

$$\|f_n(x) - f(x), g(y)\| \leq \sum_{i=1}^k \sum_{j=1}^t |a_i b_j| \cdot \|f_n(x_i) - f(x_i), g(y_j)\|.$$

Because $\lim_{n \rightarrow \infty} \|f_n(x_i) - f(x_i), g(y_j)\| = 0$ for each $x_i, y_j \in A$,

then $\lim_{n \rightarrow \infty} \|f_n(x) - f(x), g(y)\| = 0$ i.e. the sequence $\{f_n : n \in N\}$ is convergent to f on X_0 . Let $\|f_n, g\| \leq M$ for every $n \in N$. Let us take a number $\varepsilon > 0$, $x \in X$ and $y \in X$ such that $y \neq 0$. Since X_0 is a dense set in X , we can choose $x_0 \in X_0$, $x_0 \neq 0$ such that $\|x - x_0\| < \frac{\varepsilon}{6M\|y\|}$. Moreover, there exists $y_0 \in X_0$ with the property $\|y - y_0\| < \frac{\varepsilon}{6M\|x_0\|}$. The sequence $\{f_n(x_0) : n \in N\}$ is convergent in $(Y, \tau_1(Y))$, so it is a Cauchy sequence in this space. Therefore, there exists a number n_0 such that $\|f_n(x_0) - f_m(x_0), g(y_0)\| < \frac{\varepsilon}{3}$ for each $n, m \geq n_0$. As a consequence, we obtain,

$$\begin{aligned} \|f_n(x) - f_m(x), g(y)\| &\leq \|f_n(x) - f_n(x_0), g(y)\| + \|f_n(x_0) - f_m(x_0), g(y_0)\| \\ &\quad + \|f_m(x_0) - f_m(x), g(y)\| \\ &\leq \|f_n, g\| \cdot \|x - x_0\| \cdot \|y\| + \|f_n(x_0) - f_m(x_0), g(y - y_0) + g(y_0)\| \\ &\quad + \|f_m, g\| \cdot \|x - x_0\| \cdot \|y\| \\ &\leq 2M \cdot \|x - x_0\| \cdot \|y\| + \|f_n(x_0) - f_m(x_0), g(y - y_0)\| \\ &\quad + \|f_n(x_0) - f_m(x_0), g(y_0)\| \\ &< 2M \cdot \|x - x_0\| \cdot \|y\| + \|f_n(x_0), g(y - y_0)\| \\ &\quad + \|f_m(x_0), g(y - y_0)\| + \frac{\varepsilon}{3} \\ &< 2M \cdot \frac{\varepsilon}{6M\|y\|} \cdot \|y\| + \|f_n, g\| \cdot \|x_0\| \cdot \|y - y_0\| \\ &\quad + \|f_m, g\| \cdot \|x_0\| \cdot \|y - y_0\| \\ &< \frac{2}{3}\varepsilon + 2M\|x_0\| \cdot \frac{\varepsilon}{6M\|x_0\|} = \varepsilon \quad \text{for } n, m \geq n_0. \end{aligned}$$

If $y = 0$, then the inequality $\|f_n(x) - f_m(x), g(y)\| = 0 < \varepsilon$ is also true. Hence we have shown that $\{f_n(x) : n \in N\}$ is a Cauchy sequence in $(Y, \tau_1(Y))$ for every $x \in X$. Because $(Y, \tau_1(Y))$ is a sequentially complete space, then the sequence $\{f_n : n \in N\}$ is pointwise convergent. Let us denote $h(x) = \lim_{n \rightarrow \infty} f_n(x)$ for every $x \in X$. The fact that $(Y, \tau_1(Y))$ is a Hausdorff space implies $h(x) = f(x)$ for $x \in A$, i.e. $(h - f)(x)$ for $x \in A$. The operator $h - f$ is linear, thus $(h - f)(x) = 0$ for every $x \in X_0$. Using theorem 3, we see that $h \in \theta^g$. Because θ^g is a linear subspace, then $h - f \in \theta^g$. Thus there exists a positive number K such that

$$\|(h - f)(x), g(y)\| \leq K \cdot \|x\| \cdot \|y\| \quad \text{for every } x, y \in X.$$

Let $\varepsilon > 0$, $x, y \in X$, $y \neq 0$. Since the set X_0 is dense in X , we can choose $x_0 \in X_0$ such that $\|x - x_0\| < \frac{\varepsilon}{K \cdot \|y\|}$. Then

$$\begin{aligned} 0 &\leq \|(h - f)(x), g(y)\| = \|(h - f)(x - x_0) + (h - f)(x_0), g(y)\| \\ &= \|(h - f)(x - x_0), g(y)\| \\ &\leq K \cdot \|x - x_0\| \cdot \|y\| < \varepsilon. \end{aligned}$$

This gives $\|(h - f)(x), g(y)\| = 0$ for each $x \in X$, $y \in X \setminus \{0\}$. Thus $h(x) = f(x)$ for every $x \in X$. As a consequence we have shown that the sequence $\{f_n : n \in N\}$ is pointwise convergent to f , which finishes the proof. ■

Theorem 5. *Let $(X, \|\cdot\|)$ be a Banach space, $(Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space and $\{(f_n, g_n)\}_{n \geq 1}$ a sequence of elements from Θ . Then the following conditions are equivalent:*

- (a) *The sequence of 2-norms $\{\|f_n, g_n\|\}_{n \geq 1}$ is bounded.*
- (b) $\exists M > 0 \forall x, y \in X, \|x\| \leq 1, \|y\| \leq 1, \forall n \geq 1, \|f_n(x), g_n(y)\| \leq M.$
- (c) *The following conditions are satisfied:*
- (i) $\forall x \in X, \exists M_x > 0, \forall y \in X, \|y\| \leq 1, \forall n \geq 1, \|f_n(x), g_n(y)\| \leq M_x.$
- (ii) $\forall y \in X, \exists M_y > 0, \forall x \in X, \|x\| \leq 1, \forall n \geq 1, \|f_n(x), g_n(y)\| \leq M_y.$

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