Variation Method of Solving the Irregular Oblique Derivative Problem for Elliptic Equations of Second Order¹

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Abstract

This paper deals with irregular oblique derivative boundary value problems for nonlinear elliptic equations of second order. The uniformly elliptic equations of second order are reduced to elliptic complex systems of first order, and then the irregular oblique derivative problem for the equations of second order is transformed into the Riemman-Hilbert problem for the complex system of first order. The approximate solutions of the boundary value problem are obtained by the variation

¹Project supported by the National Natural Science Foundation of China(No.10671207).

method. The error estimates for the approximate solutions are also derived.

Mathematics Subject Classification: 35J60; 35J65

Keywords: variation method, irregular oblique derivative problem, elliptic complex equations, error estimates

1 Transformation of elliptic equations and formulation of the problem

We consider the quasilinear uniformly elliptic equation of second order

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + au_x + bu_y + cu = d, \quad (1)$$

where the coefficients a_{jk} ($j, k = 1, 2$), a, b, c, d are known bounded real measurable functions of $(x, y) \in \overline{D}$ and $u, u_x, u_y \in \mathbf{R}$. Here D is a bounded domain in $\mathbf{R}^2$. Under certain conditions, the above equation can be transformed into the complex form

$$u_{z\bar{z}} = F(z, u, u_z, u_{z\bar{z}}), \quad F = \operatorname{Re}[Qu_{z\bar{z}} + A_1u_z + A_2u + A_3], \quad (2)$$

where $Q = Q(z, u, u_z)$, $A_j = A_j(z, u, u_z)$ ($j = 1, 2, 3$) with

$$z = x + iy, \quad u_z = \frac{1}{2}[u_x - iu_y] = \overline{u_{\bar{z}}}, \quad u_{z\bar{z}} = \frac{1}{4}[u_{xx} + u_{yy}], \quad u_{zz} = \frac{1}{4}[u_{xx} - u_{yy} + 2iu_{xy}].$$

In fact, from (1), we have

$$Q = \frac{-a_{11} + a_{22} - 2a_{12}i}{a_{11} + a_{22}}, \quad A_1 = \frac{-a - bi}{a_{11} + a_{22}}, \quad A_2 = \frac{-c}{2(a_{11} + a_{22})}, \quad A_3 = \frac{d}{2(a_{11} + a_{22})}.$$

Correspondingly, the uniformly elliptic condition

$$\Delta = a_{11}a_{22} - a_{12}^2 \geq \Delta_0 > 0, \quad a_{11} > 0 \quad \text{in } \overline{D}$$

is reduced to the inequality

$$|Q(z, u, u_z)| \leq q_0 < 1 \quad \text{in } \overline{D},$$

in which Δ_0, q_0 are non-negative constants. Suppose that equation (2) satisfies the following conditions.

Condition C 1) $Q(z, u, u_z), A_j(z, u, u_z)$ ($j = 1, 2, 3$) are continuous in $u \in \mathbf{R}, u_z \in \mathbf{C}$ for almost every $z \in D$, and $Q = 0, A_j = 0$ ($j = 1, 2, 3$) for $z \notin D$.

2) The above functions are continuous in $z \in D$ for all continuously differentiable functions $u(z)$ in $\overline{D}$, and satisfy

$$Q, A_j (j = 1, 2, 3) \in W_p^1(D), \quad (3)$$

in which $p(> 2)$ is a positive constant.

3) For $u_j(z) \in W_{p_0}^2(D)$ ($j = 1, 2$), the function $F(z, u, u_z, u_{zz})$ satisfies the condition

$$F(z, u_1, u_{1z}, u_{1zz}) - F(z, u_2, u_{2z}, u_{2zz}) = \operatorname{Re}[\tilde{Q}(u_1 - u_2)_{zz} + \tilde{A}_1(u_1 - u_2)_z + \tilde{A}_2(u_1 - u_2)], \quad (4)$$

where $\tilde{Q}, \tilde{A}_j$ ($j = 1, 2, 3$) satisfy the conditions as those of Q, A_j ($j = 1, 2$), $p_0(2 < p_0 \leq p)$ is a positive constant.

Let D be an $(N+1)$ -connected domain with the boundary $\Gamma = \bigcup_{j=0}^N \Gamma_j \in C^2$, where Γ_j ($j = 1, \dots, N$) are located in the domain bounded by Γ_0 . Without loss of generality, we assume that D is a circular domain in $|z| < 1$ with the boundary $\Gamma_j = \{|z - z_j| = r_j\}$ ($j = 0, 1, \dots, N, \Gamma_0 = \Gamma_{N+1} = |z| = 1$) and $0 \in D$.

Problem P Find a continuously differentiable solution $u(z)$ of (2) in $\overline{D}$, which satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} = 2r(z) + h(z), z \in \Gamma, \quad (5)$$

where $\nu (= \nu_1 + i\nu_2)$ can be arbitrary at every point on Γ .

It is easy to see that the boundary condition (5) can be rewritten in the complex form

$$\operatorname{Re}[\overline{\lambda(z)}u_z] = r(z) + h(z), z \in \Gamma, \quad (6)$$

in which $\lambda(z) = \cos(\nu, x) + i \sin(\nu, x) = e^{i(\nu, x)} \neq 0$ for the angle (ν, x) between ν and the x -axis. We suppose that $\lambda(z), r(z)$ satisfy the conditions

$$C^1[\lambda(z), \Gamma] \leq k_0, C^1[r(z), \Gamma] \leq k_1, \quad (7)$$

in which k_0, k_1 are non-negative constants. The function $h(z)$ in (6) has the following form

$$h(z) = \begin{cases} 0, z \in \Gamma, & \text{if } K \geq N, \\ \left. \begin{aligned} &h_j, z \in \Gamma_j, j = 1, \dots, N - K, \\ &0, z \in \Gamma_j, j = N - K + 1, \dots, N + 1, \end{aligned} \right\} & \text{if } 0 \leq K < N, \\ \left. \begin{aligned} &h_j, z \in \Gamma_j, j = 1, \dots, N, \\ &h_j \eta_j(z), z \in \Gamma_0^j, j = N + 1, \dots, N - 2K - 1, \end{aligned} \right\} & \text{if } K < 0, \end{cases} \quad (8)$$

where $K = \Delta_\Gamma \arg \lambda(z)/2\pi$ is the index, Γ_0^j ($j = N+1, \dots, N-2K-1$) are non-degenerate mutually disjoint arcs of Γ_0 , each of which consists of inner points of Γ_0^j , h_j ($j = 1, \dots, N-2K-1$) are unknown real constants to be determined appropriately, and $\eta_j(z)$ is a positive continuous function on Γ_0^j such that $\eta_j(z) = 0$ on $\Gamma_0 \setminus \Gamma_0^j$ and $C_\alpha[\eta_j, \Gamma_0] \leq k_0$ ($j = N+1, \dots, N-2K-1$) (see Chapter I, [5]). Moreover $w(z) = u_z$ satisfies the point conditions

$$L_j w = \operatorname{Im}[\overline{\lambda(a_j)} w(a_j)] = b_j, \quad j \in J = \begin{cases} 1, \dots, 2K-N+1, & K \geq N, \\ N-K+1, \dots, N+1, & 0 \leq K < N, \end{cases} \quad (9)$$

where $a_j \in \Gamma_j$ ($j = 1, \dots, N$), $a_j \in \Gamma_0$ ($j = 0, N+1, \dots, 2K-N+1, K \geq N$) are distinct points, b_j ($j \in J$) are real constants. The functions $u(z)$, $w(z)$ satisfy the relation

$$u(z) = 2\operatorname{Re} \int_{a_0}^z w(z) dz + b_0, \quad (10)$$

where b_0 is a real constant.

By the method in [2] and [3], we see that Problem P for (2) can be reduced to the following Riemann-Hilbert boundary value problem.

Problem A Find a solution $[u(z), w(z)]$ of the system of first order equations

$$w_{\bar{z}} = F(z, u, w, w_z), \quad u_{\bar{z}} = \overline{w(z)} \quad \text{in } D, \quad (11)$$

such that $[u(z), w(z)]$ satisfies the boundary conditions

$$\operatorname{Re}[\overline{\lambda(z)} w(z)] = r(z) + h(z), \quad \operatorname{Re}[iu(z)] = 0 \quad \text{on } \Gamma, \quad L_j w = b_j, \quad j \in J + \{0\}, \quad (12)$$

where $L_0 w = \operatorname{Re} u(a_0) = b_0$ for $a_0 = 1$ and a real constant b_0 . In fact, if $[u(z), w(z)]$ is the solution of Problem A for the system (11) in D , the substitution shows that $u(z)$ satisfies the equation

$$[\operatorname{Im} u(z)]_{z\bar{z}} = \operatorname{Im} F(z, u, w, w_z) = 0 \quad \text{in } D,$$

and the boundary conditions

$$\operatorname{Re}[iu(z)] = 0, \quad \text{and} \quad \operatorname{Re}[z(\operatorname{Im} u(z))_z] = 0 \quad \text{on } \Gamma.$$

The index of the above boundary value problem is $N-1$, namely $\Delta[\arg \bar{z}]_\Gamma = N-1$. From the extremum principle of solutions of harmonic equation, we have $\operatorname{Im} u(z) = 0$ in $\overline{D}$, hence $u(z) = \operatorname{Re} u(z)$ in $\overline{D}$.

In [4], L.S. Klabukova and I.I. Chechel discussed the approximate solution of the Riemann-Hilbert problem for generalized analytic function using the variation-difference method, where the generalized analytic function was defined as a solution of the complex equation $W_{\bar{z}} = A(z)W + B(z)\overline{W}$ in D . Since their modified boundary value problem involves the conjugate complex

equation, it is difficult to generalize the related result to the case of general linear uniformly elliptic complex equations of first order and equations of second order. In the following, applying the proposed method in [2] which is different from that in [4], we can discuss the general boundary value problem (Problem P) for (2) and find an approximate solution for Problem P .

2 Variation method of solving Problem A

Problem B Find $U(z), W(z) \in W_p^1(D)$ ($p > 2$), such that

$$\begin{aligned} I(U, W) = & \min_{u, w \in W_p^1(D)} I(u, w) = \min_{u, w \in W_p^1(D)} \left\{ \iint_D |w_{\bar{z}} - F(z, u, w, w_z)|^2 d\sigma_z \right. \\ & + \iint_D |u_{\bar{z}} - \overline{w(z)}|^2 d\sigma_z + \int_{\Gamma} |\operatorname{Re}[\overline{\lambda(t)}w(t)] - r(t) - h(t)|^2 ds \\ & \left. + \int_{\Gamma} |\operatorname{Re}[iu(t)]|^2 ds + \sum_{j \in J + \{0\}} |L_j w - b_j|^2 \right\}, \end{aligned} \quad (13)$$

where $\lambda(z), r(z), L_j w, b_j (j \in J + \{0\})$ are as stated in (6), (9) and (12), and $h(t)$ is an undetermined function depending on $w(z)$ (Henceforth we may denote $h(t)$ by $h(t, w)$).

Theorem 1 *The system of functions $[U(z), W(z)]$ is a solution of Problem A for (11) if and only if it is a solution of Problem B.*

Proof Substitute the solution $[U(z), W(z)]$ of Problem A into the appropriate positions of u, w in (13), we have $I(U, W) = 0$. This shows that $[U(z), W(z)]$ is a solution of Problem B. Conversely, if $[U(z), W(z)]$ is a solution of Problem B, it is clear that $[U(z), W(z)]$ satisfies the first order complex system in (11) for almost every point $z \in D$ and the conditions in (12). If we can prove that $U(z), W(z) \in W_{p_0}^1(D)$, $2 < p_0 \leq p$, then $[U(z), W(z)]$ is also a solution of Problem A. In fact, substitute the solution $[U(z), W(z)]$ into the complex system (11) and denote $Q_0(z) = [Q(z) + \overline{Q(z)}\overline{W_{\bar{z}}}/W_z]/2$, $f(z) = A_1(z)W + A_2(z)U + A_3(z)$. Then (2) can be rewritten in the form

$$W_{\bar{z}} - Q(z)W_z = f(z). \quad (14)$$

From [1] and [2], the boundary value problem (6), (9), (10), (14) has a unique solution $\phi(z) \in W_{p_0}^1(D)$, and $w(z) = W(z) - \phi(z)$ is a solution of the following homogeneous boundary value problem

$$w_{\bar{z}} - Q(z)w_z = 0, \quad u_{\bar{z}} - \overline{w(z)} = 0, \quad z \in D, \quad (15)$$

$$\operatorname{Re}[\overline{\lambda(z)}w(z)] = H(z), \quad \operatorname{Re}[iu(z)] = 0, \quad z \in \Gamma, \quad L_j w = 0, \quad j \in J + \{0\},$$

where $H(z)$ is an undetermined function. As stated in [1] and [2], the Beltrami equation in (15) has a homeomorphic solution $\zeta(z) \in W_{p_0}^1(D)$ which quasiconformally maps D onto an $(N + 1)$ -connected circular domain Ω in the unit

disk $|\zeta| < 1$, and $w(z)$ can be expressed as $w(z) = \Phi[\zeta(z)]$, in which $\Phi(\zeta)$ is analytic in Ω and continuous in $\overline{\Omega}$, and satisfies the boundary conditions

$$\begin{aligned} \operatorname{Re}[\overline{\lambda[z(\zeta)]}][\Phi(\zeta) + \phi(z(\zeta))] &= H[z(\zeta)], \quad \zeta \in \zeta(\Gamma), \\ \operatorname{Re}[iu(z(\zeta))] &= 0, \quad \zeta \in \zeta(\Gamma); \quad L_j w = 0, \quad j \in J + \{0\}. \end{aligned}$$

Here $z(\zeta)$ is the inverse function of $\zeta(z)$. Moreover, we can derive that $\Phi(\zeta) \in C_\beta(\overline{\Omega})$, $0 < \beta \leq \alpha$. Hence $w(z) \in W_{p_0}^1(D)$. From $U(z) = \int_{a_0}^z W(z)dz + b_0$, it follows that $U(z) \in W_{p_0}^1(D)$.

On the basis of (13), we see that if $U = u, W = w$, the variation of $I(u, w)$ is equal to zero, namely

$$\begin{aligned} \delta I(u, w) &= \\ &= 2\operatorname{Re} \int \int_D \{[w_z - F(z, u, w, w_z)][\tilde{w}_z - \overline{F_*(z, \tilde{u}, \tilde{w}, \tilde{w}_z)}] + [u_z - \overline{w}][\tilde{u}_z - \tilde{w}]\} d\sigma_z \\ &+ 2 \int_\Gamma \{[\operatorname{Re} \overline{\lambda(t)} w(t) - r(t) - h(t, w)][\operatorname{Re} \overline{\lambda(t)} \tilde{w}(t) - h(t, \tilde{w})] - \operatorname{Re}[iu] \operatorname{Re}[i\tilde{u}]\} ds \\ &+ 2 \sum_{j \in J + \{0\}} [L_j w - b_j] L_j(\tilde{w}) = 0, \end{aligned} \tag{16}$$

in which $\tilde{u} = \delta u, \tilde{w} = \delta w$ and $F_*(z, \tilde{u}, \tilde{w}, \tilde{w}_z) = \operatorname{Re}[Q(z)\tilde{w}_z + A_1\tilde{w}] + A_2\tilde{u}$. Setting

$$\begin{aligned} f(u, \tilde{u}, w, \tilde{w}) &= \\ &= 2\operatorname{Re} \int \int_D \{[w_z - F_*(z, u, w, w_z)][\tilde{w}_z - \overline{F_*(z, \tilde{u}, \tilde{w}, \tilde{w}_z)}] + [u_z - \overline{w}][\tilde{u}_z - \tilde{w}]\} d\sigma_z \\ &+ 2 \int_\Gamma \{[\operatorname{Re} \overline{\lambda(t)} w(t) - h(t, w)][\operatorname{Re} \overline{\lambda(t)} \tilde{w}(t) - h(t, \tilde{w})] - \operatorname{Re}[iu] \operatorname{Re}[i\tilde{u}]\} ds \\ &+ 2 \sum_{j \in J + \{0\}} L_j w L_j \tilde{w}, \\ g(\tilde{u}, \tilde{w}) &= 2\operatorname{Re} \int \int_D A_3(z)[\tilde{u}_z - \overline{F_*(z, \tilde{u}, \tilde{w}, \tilde{w}_z)}] d\sigma_z \\ &+ 2\operatorname{Re} \int_\Gamma r(t)[\operatorname{Re} \overline{\lambda(t)} \tilde{w}(t) - h(t, \tilde{w})] ds + 2 \sum_{j \in J + \{0\}} b_j L_j \tilde{w}, \end{aligned}$$

the formula (16) leads to

$$f(u, \tilde{u}, w, \tilde{w}) = g(\tilde{u}, \tilde{w}). \tag{17}$$

Problem C Find a system of complex functions: $u(z), w(z) \in H^*(D) = H^1(D) \cap C(\overline{D})$, such that the equation (17) holds for any functions $\tilde{u}(z), \tilde{w}(z) \in H^*(D)$.

Theorem 2 *A necessary and sufficient condition that $[u(z), w(z)]$ is a solution of Problem B is that it is a solution of Problem C.*

Proof It is evident that the necessity is true. Now we prove the sufficiency. Let $[u(z), w(z)]$ be a solution of Problem A (the existence and uniqueness of the solution for Problem A have been proved in [2]). By Theorem 1 and the necessity of this theorem, the solution $[u(z), w(z)]$ is also a solution of Problem B and Problem C. If the solution of Problem C is unique, then the sufficiency in this theorem is proved. In fact, let $[u^*(z), w^*(z)]$ be another solution of Problem C, we have

$$f(u^*, \tilde{u}, w^*, \tilde{w}) = g(\tilde{u}, \tilde{w}).$$

From (17), it follows that $f(u^*, \tilde{u}, w^*, \tilde{w}) - f(u, \tilde{u}, w, \tilde{w}) = f(u^* - u, \tilde{u}, w^* - w, \tilde{w}) = 0$, where $\tilde{u}(z) = u^*(z) - u(z)$, $\tilde{w}(z) = w^*(z) - w(z)$. Therefore $[\tilde{u}(z), \tilde{w}(z)]$ is a solution of the following homogeneous boundary value problem

$$\begin{aligned} \tilde{w}_{\bar{z}} &= F(z, \tilde{u}, \tilde{w}, \tilde{w}_z) - A_3(z), \quad \tilde{u}_{\bar{z}} = \overline{\tilde{w}(z)}, \quad z \in D, \\ \operatorname{Re}[\overline{\lambda(t)}\tilde{w}(t)] &= h(t), \quad t \in \Gamma; \quad \operatorname{Re}[i\tilde{u}(t)] = 0, \quad t \in L, \\ L_j\tilde{w} &= 0, \quad j \in J + \{0\}, \end{aligned}$$

and then $\tilde{u}(z) = 0$, $\tilde{w}(z) = 0$ in D . This shows the uniqueness of Problem C.

By Theorems 1 and 2, we can see that if $[u(z), w(z)]$ is a solution of Problem C, then it is also a solution of Problems B and A.

Next we shall establish the algebraic equation corresponding to Problem B by using difference methods. Cut the domain D by the square net with the scale $h_j \leq h$, where h is a sufficient small positive number. Then the plane is divided into small square domains. We require that the square net includes the $a_j (j \in J + \{0\})$. Moreover every small square domain $\Delta[jk] ((j, k) = (x_j, y_k))$ is subdivided by the diagonal line segment joining $(x_j + h_j, y_k)$ and $(x_j, y_k + h_j)$ into two triangles, i.e. the upper triangle $\Delta^+[jk]$ and the lower triangle $\Delta^-[jk]$. Denote by $\overline{D}_h$ the sum of all triangles such that $\Delta^\pm[jk] \cap \overline{D} \neq \emptyset$, and by Γ_h the sum of boundary triangles of $\overline{D}_h$ with the boundary nodes. Obviously $\overline{D} \subset \overline{D}_h$. We define the function $\tilde{W}_h(x, y) = \tilde{W}_h(z)$. For example, in the triangle $\Delta^-[jk]$ with the vertex node $(j, k) = (x_j, y_k)$, we define

$$\tilde{W}_h(x, y) = \tilde{W}_{j,k} + \frac{\tilde{W}_{j+1,k} - \tilde{W}_{j,k}}{h_j}(x - x_j) + \frac{\tilde{W}_{j,k+1} - \tilde{W}_{j,k}}{h_j}(y - y_j), \quad (18)$$

where $\tilde{W}_{j,k}$, $\tilde{W}_{j+1,k}$ and $\tilde{W}_{j,k+1}$ are the values of $\tilde{W}_h$ at three vertexes of $\Delta^-[jk]$ respectively. The values of $\tilde{W}_h$ at other nodes can be similarly defined. It is clear that the function $\tilde{W}_h(x, y)$ based on the above method is linearly continuous in $\overline{D}_h$. Similarly we can discuss the function $\tilde{U}_h(x, y) = \tilde{U}_h(z)$. Thus we can find the minimum of the functional $J(\tilde{U}_h, \tilde{W}_h)$.

Problem D Find $\tilde{u}_h(z), \tilde{w}_h(z) \in W_p^1(D) (p > 2)$ such that

$$\begin{aligned} J(\tilde{u}_h, \tilde{w}_h) &= \\ &= \min_{\tilde{U}_h, \tilde{W}_h \in W_p^1(D)} J(\tilde{U}_h, \tilde{W}_h), \quad J(\tilde{U}_h, \tilde{W}_h) = \iint_D \{ |\tilde{W}_{h\bar{z}} - F(z, \tilde{U}_h, \tilde{W}_h, \tilde{W}_{hz})|^2 \\ &\quad + |\tilde{U}_{h\bar{z}} - \overline{\tilde{W}_h}|^2 \} d\sigma_z + \int_\Gamma \{ |\operatorname{Re}[\overline{\lambda(t)} \tilde{W}_h] - r(t) - h(t, \tilde{W}_h)|^2 + |\operatorname{Re}[i\tilde{U}_h]|^2 \} ds, \\ L_j \tilde{W}_h &= b_j, j \in J + \{0\} \end{aligned} \quad (19)$$

For convenience we only discuss the case of the function $\tilde{W}_h(x, y)$ and $K \geq N$. The other cases can be similarly discussed. We shall prove that there exists the solution $[\tilde{u}_h, \tilde{w}_h]$ of Problem D. In fact, letting m, n be the totals of nodes belonging to $\Gamma_h, \overline{D}_h$ respectively, and $\tilde{W}_k (k = 1, \dots, m), \tilde{W}_l (l = 1, \dots, n)$ be the values of $\tilde{W}_h$ at the nodes $z_k \in \Gamma_h$ and $z_l \in \overline{D}_h$ respectively, then from (17) and (19), we can derive (see [4])

$$J(\tilde{W}_h) = J(\tilde{W}_1, \dots, \tilde{W}_n) = \sum_{j,k=1}^n a_{jk} \tilde{W}_j \tilde{W}_k + 2 \sum_{j=1}^n c_j \tilde{W}_j + d, \quad (20)$$

with the conditions

$$\sum_{k=1}^m b_{lk} \tilde{W}_k = b_l, \quad l = 1, \dots, 2K - N + 1, \quad (21)$$

in which $b_l (l = 1, \dots, 2K - N + 1, K' = 2K - N + 1)$ are known, and a_{jk}, c_j, d, b_{lk} can be found. It is not difficult to see that (20) and (21) can be solved with the aid of the Lagrange multipliers $\lambda_l (l = 1, \dots, K')$, i.e. there exist real constants $\lambda_l (l = 1, \dots, K')$, such that $J(\tilde{W}_1, \dots, \tilde{W}_n) + 2 \sum_{l=1}^{K'} \lambda_l (\sum_{k=1}^m b_{lk} \tilde{W}_k - b_l)$ attains the minimum. For this we obtain the derivatives of (20) with respect to $\tilde{W}_j$ such that they are equal to zero, and noting that $a_{jk} = a_{kj}$, we have

$$\begin{aligned} \sum_{k=1}^n a_{jk} W_k + \sum_{l=1}^{K'} \lambda_l b_{lj} + c_j &= 0, \quad j = 1, \dots, m, \\ \sum_{k=1}^n a_{jk} W_k + c_j &= 0, \quad j = m+1, \dots, n; \quad \sum_{k=1}^m b_{lk} W_k = b_l, \quad l = 1, \dots, K'. \end{aligned} \quad (22)$$

The above linear system of algebraic equations is solvable, because the corresponding homogeneous system

$$\begin{aligned} \sum_{k=1}^n a_{jk} W_k + \sum_{l=1}^{K'} \lambda_l b_{lj} &= 0, \quad j = 1, \dots, m, \\ \sum_{k=1}^n a_{jk} W_k &= 0, \quad j = m+1, \dots, n, \quad \sum_{k=1}^m b_{lk} W_k = 0, \quad l = 1, \dots, K' \end{aligned} \quad (23)$$

has only the zero solution. In fact let $(W_1^0, \dots, W_n^0)$ be a solution of the system (23). Then we have $\sum_{l=1}^{K'} \lambda_l \sum_{k=1}^m b_{lk} W_k^0 = 0$, and the solution satisfies the equality

$$J(W_1^0, \dots, W_n^0) + \sum_{l=1}^{K'} \lambda_l \sum_{k=1}^m b_{lk} W_k^0 = \iint_D |\tilde{W}_{h\bar{z}}^0 - F(z, \tilde{U}_h^0, \tilde{W}_h^0, \tilde{W}_{hz}^0) + A_3(z)|^2 d\sigma_z + \int_{\Gamma} |\operatorname{Re} \bar{\lambda}(t) \tilde{W}_h^0 - h(t, \tilde{W}_h^0)|^2 ds = 0. \quad (24)$$

By Theorems 1 and 2, it is also the approximate solution of the corresponding homogeneous problem of Problem A. On the basis of the uniqueness of Problem A, it can be derived that $U_k^0 = 0, W_k^0 = 0, k = 1, \dots, n$ (see [2]).

Setting that $\mathbf{W} = (W_1, \dots, W_n)^T, \mathbf{A}_1 = (a_{jk})_{m \times n}, \mathbf{A}_2 = (a_{jk})_{(n-m) \times n}, \lambda = (\lambda_1, \dots, \lambda_{K'})^T, \mathbf{b} = (b_1, \dots, b_{K'})^T, \mathbf{B} = (b_{lj})_{K' \times m}, \mathbf{c}_1 = (c_1, \dots, c_m)^T, \mathbf{c}_2 = (c_{m+1}, \dots, c_n)^T$, the system (22) can be rewritten as

$$\mathbf{A}_1 \mathbf{W} + \mathbf{B}^T \lambda + \mathbf{c}_1 = 0, \quad \mathbf{A}_2 \mathbf{W} + \mathbf{c}_2 = 0, \quad \mathbf{B} \mathbf{W} + \mathbf{b} = 0. \quad (25)$$

From the first formula in (25), $\mathbf{B} \mathbf{A}_1 \mathbf{W} + \mathbf{B} \mathbf{B}^T \lambda + \mathbf{B} \mathbf{c}_1 = 0$, we see that the matrix $\mathbf{B} \mathbf{B}^T$ is non-degenerate, and can obtain

$$(\mathbf{E} - \mathbf{B}^T (\mathbf{B} \mathbf{B}^T)^{-1} \mathbf{B}) (\mathbf{A}_1 \mathbf{W} + \mathbf{c}_1) = 0, \quad \mathbf{A}_2 \mathbf{W} + \mathbf{c}_2 = 0, \quad (26)$$

$$\mathbf{B} \mathbf{W} + \mathbf{b} = 0, \quad (27)$$

where $\mathbf{E}$ is unit matrix. Thus we have the following result: (i) The system (26) with arbitrary $\mathbf{c} = (c_1, \dots, c_n)^T$ is solvable, which is corresponding to the solvability of Problem A. (ii) The homogeneous problem corresponding to (26) has $2K - N + 1$ ($K \geq N$) linearly independent solutions. (iii) The solution of (26) is the solution of Problem D and also is the approximate solution of Problem A. Hence we have the following theorem.

Theorem 3 Under the above Condition C, (9) and (10), there exists a solution of Problem D.

Below we present the error estimate of approximate solutions. From (6), (9), (10) and (19), we have

$$\begin{aligned} J(\tilde{u}_h - u, \tilde{w}_h - w) &= \min_{\tilde{U}_h, \tilde{W}_h \in W_p^1(D)} J(\tilde{U}_h - u, \tilde{W}_h - w), J(\tilde{U}_h - u, \tilde{W}_h - w) \\ &= \iint_D \{ |(\tilde{W}_h - w)_{\bar{z}} - F(z, \tilde{U}_h - u, \tilde{W}_h - w, (\tilde{W}_h - w)_z)|^2 \\ &\quad + |(\tilde{U}_h - u)_{\bar{z}} - (\overline{\tilde{W}_h - w})|^2 \} d\sigma_z \\ &\quad + \int_{\Gamma} \{ |\operatorname{Re}[\bar{\lambda}(t)(\tilde{W}_h - w)] - h(t, \tilde{W}_h - w)|^2 \\ &\quad + |\operatorname{Re}[i(\tilde{u}_h - u)]|^2 \} ds, \quad L_j(\tilde{W}_h - w) = 0, \quad j \in J + \{0\}. \end{aligned} \quad (28)$$

It is clear that the general solution of the boundary value problem (2),(6),(9), called Problem $\tilde{A}$, can be expressed as

$$w(z) = w_0(z) + \sum_{k=m'}^{m''} c_k w_k(z), \quad u(z) = 2\operatorname{Re} \int_0^z w(z) dz + b_0,$$

$$m' = \begin{cases} 1, & K \geq N, \\ N-K+1, & 0 \leq K < N, \\ \emptyset, & K < 0, \end{cases} \quad m'' = \begin{cases} 2K-N+1, & K \geq N, \\ N+1, & 0 \leq K < N, \\ \emptyset, & K < 0, \end{cases}$$

where $w_0(z)$ is a special solution of Problem $\tilde{A}$ and $w_k(z) (k = m', \dots, m'')$ are the complete system of linearly independent solutions, and $c_k (k = m', \dots, m'')$ are arbitrary real constants. If the solution $[u(z), w(z)]$ satisfies the point condition (9), i.e. $[u(z), w(z)]$ is the solution of Problem A, then the constants $c_k (k = m', \dots, m'')$ are determined. For every function $w_k(z) (m' \leq k \leq m'')$, there exists the corresponding function $\tilde{w}_k(z)$, according to the above method, that is an approximation of $w_k(z)$ and is continuous in $\overline{D}$. Similarly to the preceding discussion, we have

$$\tilde{w}(z) = \tilde{w}_0(z) + \sum_{k=m'}^{m''} \tilde{c}_k \tilde{w}_k(z), \quad \tilde{u}(z) = 2\operatorname{Re} \int_0^z \tilde{w}(z) dz + b_0, \quad (29)$$

in which $\tilde{c}_k (k = m', \dots, m'')$ are chosen real constants such that $\tilde{L}_j(\bar{\lambda} \tilde{w}) = L_j[\tilde{w}_0 + \sum_{k=m'}^{m''} \tilde{c}_k \tilde{w}_k(z)] = b_j, j = m', \dots, m''$. Obviously for the smooth function $w_k(z)$

$$\max_{\overline{D}} |w_k(z) - \tilde{w}_k(z)| = O(h^2), \quad \max_{\overline{D}} \left| \frac{\partial}{\partial \bar{z}} [w_k(z) - \tilde{w}_k(z)] \right| = O(h),$$

we have $|c_k - \tilde{c}_k| = O(h^2), k = m', \dots, m'', \max_{\overline{D}_h} |w(z) - \tilde{w}(z)| = O(h^2), \max_{\overline{D}_h} \left| \frac{\partial}{\partial \bar{z}} [w(z) - \tilde{w}(z)] \right| = O(h)$. Moreover $J(\tilde{w}) = O(h^2)$ can be derived. Noting that $\tilde{w}(z)$ satisfies the condition (9) or (28),

$$\begin{aligned} & \int \int_D \{ |(\tilde{w}_h - w)_{\bar{z}} - F(z, \tilde{u}_h - u, \tilde{w}_h - w, (\tilde{w}_h - w)_z)|^2 + |(\tilde{u}_h - u)_{\bar{z}} - \overline{\tilde{w}_h} + \overline{w}|^2 \} d\sigma_z \\ & + \int_{\Gamma} \{ |\operatorname{Re}[\overline{\lambda(t)}(\tilde{w}_h - w)] - h(t, \tilde{w}_h - w)|^2 + |\operatorname{Re}[i(\tilde{u}_h - u)]|^2 \} ds \\ & = J(\tilde{u}_h, \tilde{w}_h) \leq J(\tilde{u}, \tilde{w}) = O(h^2) \end{aligned} \quad (30)$$

is derived. Hence

$$\begin{aligned} \int \int_D \{|\tilde{w}_h - w|^2 + |\tilde{u}_h - u|^2\} d\sigma_z &\leq M \left\{ \int \int_D |(\tilde{w}_h - w)_{\bar{z}} \right. \\ &\quad \left. - F(z, \tilde{u}_h - u, \tilde{w}_h - w, (\tilde{w}_h - w)_z) \right|^2 + |(\tilde{u}_h - u)_{\bar{z}} - \overline{\tilde{w}_h} + \overline{w}|^2 \} d\sigma_z \\ &\quad + \int_{\Gamma} \{ |\operatorname{Re}[\overline{\lambda(t)}(\tilde{w}_h - w)] - h(t, \tilde{w}_h - w) |^2 + |\operatorname{Re}[i(\tilde{u}_h - u)]|^2 \} ds = O(h^2), \end{aligned} \quad (31)$$

where M is a positive constant. This is an error estimate of the approximate solution of Problem A. We summarize the above result into the following theorem.

Theorem 4 *Let $[u_h(z), w_h(z)]$ and $[u(z), w(z)]$ be the approximate and exact solutions of Problem A, respectively. Then the error estimate possesses the form (31).*

In addition, we mention that by using the method as stated in [5], more accurate error estimates of the approximate solution of Problem B can be obtained as follows:

$$\|u(z) - u_h(z)\|_{L_2(\bar{D})} = O(h^2), \quad \|w(z) - w_h(z)\|_{L_2(\bar{D})} = O(h^2).$$

3 Variation method of solving the nonlinear irregular oblique derivative problem

Consider the nonlinear uniformly elliptic equation of second order equations

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + au_x + bu_y + (c - |u|^\sigma)u = d, \quad (32)$$

where the coefficients a_{jk} ($j, k = 1, 2$), a, b, c, d are known bounded real measurable functions of $(x, y) \in \bar{D}$, $u, u_x, u_y \in \mathbf{R}$, and σ is a positive number. Here D is a bounded domain in $\mathbf{C}$. Under certain conditions, the above equation can be transformed into the complex form

$$u_{z\bar{z}} = \operatorname{Re}[Qu_{zz} + \hat{A}_1u_z + \hat{A}_2u + \hat{A}_3], \quad (33)$$

where $Q = Q(z, u, u_z)$, $\hat{A}_j = \hat{A}_j(z, u, u_z)$ ($j = 1, 2, 3$), i.e.,

$$Q = \frac{-a_{11} + a_{22} - 2a_{12}i}{a_{11} + a_{22}}, \quad \hat{A}_1 = \frac{-a - bi}{a_{11} + a_{22}}, \quad \hat{A}_2 = \frac{-c + |u|^\sigma}{2(a_{11} + a_{22})}, \quad \hat{A}_3 = \frac{d}{2(a_{11} + a_{22})},$$

and the uniformly elliptic condition

$$\Delta = a_{11}a_{22} - a_{12}^2 \geq \Delta_0 > 0, \quad a_{11} > 0 \text{ in } \bar{D}$$

is reduced to the inequality

$$|Q(z, u, u_z)| \leq q_0 < 1 \text{ in } \bar{D},$$

in which Δ_0, q_0 are non-negative constants. Suppose that equation (33) satisfies the following conditions.

Condition C' 1) $Q(z, u, u_z), \hat{A}_j(z, u, u_z) (j = 1, 2, 3)$ are continuous in $u \in \mathbf{R}, u_z \in \mathbf{C}$ for almost every $z \in D$, and $Q = 0, \hat{A}_j = 0 (j = 1, 2, 3)$ for $z \notin D$.

2) The above functions are continuous in $z \in D$ for all continuously differentiable functions $u(z)$ in $\bar{D}$, and satisfy

$$Q, \hat{A}_j (j = 1, 2, 3) \in W_p^1(D), \quad (34)$$

in which $p (> 2)$ is a positive constant.

3) The $F(z, u, u_z, u_{zz}) = \text{Re}[Qu_{zz} + \hat{A}_1u_z + \hat{A}_2u + \hat{A}_3]$ satisfies the condition similar to (4) in Section 1 and the following uniform ellipticity condition

$$|F(z, u, u_z, U_1) - F(z, u, u_z, U_2)| \leq q_0|U_1 - U_2| \text{ in } D, \quad (35)$$

for almost every point $z \in D$, any continuously differentiable function $u(z)$ in $\bar{D}$ and any $U_1, U_2 \in \mathbf{C}$.

Problem P' Find a continuously differentiable solution $u(z)$ of (32) in $\bar{D}$ which satisfies the irregular oblique derivative boundary conditions

$$\frac{\partial u}{\partial \nu} + 2d_1(z)u(z) = 2d_2(z) + h(z), z \in \Gamma, u(a_j) = b_j, j = 0, 1, \dots, K'', \quad (36)$$

where ν can be arbitrary at every point on Γ , $K'' = 2K - 2N + J + 1$, $J (\geq 0)$ are non-negative integers, $h(z)$ is as stated in (8). It is easy to see that the boundary condition (36) can be rewritten in the complex form

$$\text{Re}[\overline{\lambda(z)}u_z] + d_1(z)u = d_2(z) + h(z), z \in \Gamma, u(a_j) = b_j, j = 0, 1, \dots, K'', \quad (37)$$

in which $\lambda(z) = \cos(\nu, x) + i \sin(\nu, x) = e^{i(\nu, x)} \neq 0$ for the angle (ν, x) between ν and the x -axis, $a_j (\in \Gamma_j, j = 0, 1, \dots, K'')$ are fixed points, and $a_0 = 1$. We assume the following holds:

$$\begin{aligned} C^1[\lambda(z), \Gamma] &\leq k_0, C^1[d_1(z), \Gamma] \leq k_0, C^1[d_2(z), \Gamma] \leq k_2, \\ |b_j| &\leq k_2, j = 0, 1, \dots, K'', d_1(z) \cos(\nu, n) \geq 0 \text{ on } \Gamma, \end{aligned} \quad (38)$$

in which k_0, k_2 are non-negative constants.

Now we explain that the boundary $\partial D = \Gamma$ can be divided into two parts E^+ and E^- , where $E^+ \subset \{z \in \partial D, \cos(\nu, n) \geq 0, d_1 \geq 0\}$ and $E^- \subset \{z \in \partial D, \cos(\nu, n) \leq 0, d_1 \leq 0\}$, and $E^+ \cap E^- = \emptyset, E^+ \cup E^- = \Gamma, \overline{E^+} \cap \overline{E^-} = E^0$.

For every component $L = \Gamma_j (0 \leq j \leq N)$ of Γ , there are three possibilities: 1. $L \subset E^+$. 2. $L \subset E^-$. In these two cases, if $\cos(\nu, n) \equiv 0, d_1(z) \equiv 0$ on $\Gamma_j (1 \leq j \leq J, J \leq N+1)$, and the above identical formulas on $\Gamma_j (J < j \leq N+1)$ do not hold, then we need the conditions $\int_{\Gamma_j} d_2(z) ds = 0$ on $\Gamma_j (1 \leq j \leq J)$, and $u(a_j) = b_j, j = 0, 1, \dots, K'' (\geq J)$, in which $a_j, b_j (j = 0, 1, \dots, K'')$ are as stated before. 3. There exists at least a point on each component of $L^+ = E^+ \cap L$ and $L^- = E^- \cap L$, such that $\cos(\nu, n) \neq 0$ at the point, and $E^0 \cap L \in \{a_0, a_1, \dots, a_{K''}\}$ such that every component of L^+, L^- includes its initial point and does not include its terminal point; and $a_j \in L^+ \cap \overline{L^-} (0 \leq j \leq K'')$, when the direction of ν at a_j is equal to the direction of L ; and $a_j \in \overline{L^+} \cap L^- (0 \leq j \leq K'')$, when the direction of ν at a_j is opposite to the direction of L ; and $\cos(\nu, n)$ changes the sign once on the two components with the end point $a_j (0 \leq j \leq K'')$; we may assume that $u(a_j) = b_j, j = 0, 1, \dots, K''$, herein $a_j, b_j (j = 0, 1, \dots, K'')$ are as stated before.

The index of Problem P' is

$$K = \frac{1}{2}(K_0 + \dots + K_N) \text{ for } K_j = \Delta_{\Gamma_j} \arg \lambda(z), j = 1, \dots, N+1 \quad (39)$$

Actually the index K of $\lambda(z)$ on Γ is equal to $(m - m')/2 + N - 1$, where m, m' are the numbers of points $c_j (j = 1, \dots, m)$ and $c'_j (j = 1, \dots, m')$ respectively. Here $c_j \in E^+ \cap E^0, c'_j \in E^- \cap E^0$ when the direction of ν at c_j, c'_j is equal to the direction of Γ , $c_j \in E^- \cap E^0, c'_j \in E^+ \cap E^0$ when the direction of ν at c_j, c'_j is equal to the opposite direction of Γ .

Moreover the functions $u(z), w(z)$ satisfy the relation

$$u(z) = 2\operatorname{Re} \int_0^z w(z) dz + b_0, \quad (40)$$

where b_0 is a real constant. Similarly to Section 1, we can see that Problem P' for equation (32) is equivalent to the boundary value problem A' :

$$w_{\bar{z}} = F(z, u, w, w_z), \quad u_{\bar{z}} = \overline{w(z)} \text{ in } D, \quad (41)$$

with the boundary conditions

$$\begin{aligned} \operatorname{Re}[\overline{\lambda(z)} w(z)] + d_1(z) u &= d_2(z), \quad \operatorname{Re}[i u(z)] = 0 \text{ on } \Gamma, \\ L_j u &= u(a_j) = b_j, j = 0, 1, \dots, K''. \end{aligned} \quad (42)$$

In [6], we discuss the solvability of Problem P' for equation (33) with the special case: the index $K \geq N$ by the another complex method. Now by using the variation method similar to Sections 1 and 2, we can obtain the following result for any case of the index $K : |K| < +\infty$.

Theorem 5 *Under the above conditions, Problem P' for equation (32) has one exact solution $u(z)$ and one approximate solution $u_h(z)$ that satisfies*

$$\|u_h(z) - u(z)\|_{L_2(\bar{D})} = O(h^2).$$

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Received: November 30, 2006