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On Functorial Properties of Polynilpotent Multipliers

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Abstract

Let $\mathcal{V}$ be a variety of groups. Baer-invariant $\mathcal{V}M(-)$ is a functor from the category of all groups, $\mathcal{G}roup$, to the category of all abelian groups, $\mathcal{A}b$. In this paper, we show some functional properties of the polynilpotent multiplier of a group G of class row $(c_1, \ldots, c_t)$, $\mathcal{N}_{c_1,\ldots,c_t}M(G)$. In fact, by using some results of the author and others, we try to concentrate on the commutativity of the above functor with the two famous functors Ext and Tor.

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1. Introduction

Let G be a group with a free presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1 ,$$

where F is a free group and R a normal subgroup of F such that the above sequence is exact. Then the *Baer-invariant* of G, after R. Baer [1], with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{V(R,F)} \ ,$$

where $V(F)$ is the verbal subgroup of F with respect to $\mathcal{V}$, as

$$\{v(f_1, \cdots, \cdots, f_s) \mid v \in V, f_i \in F, 1 \leq i \leq s\},$$

and

$$V(R,F) =< v(f_1, \ldots, f_{i-1}, f_i r, f_{i+1}, \ldots, f_n) v(f_1, \ldots, f_i, \ldots f_n)^{-1} \mid r \in R,$$
$$1 \leq i \leq n, v \in V, f_i \in F, n \in \mathbf{N} > .$$

It can be proved that the Baer-invariant of a group G is independent of the choice of the presentation of G and it is always an abelian group (See [7]).

In particular, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer-invariant of G will be $(R \cap F')/[R,F]$, which, following Hopf [5], is isomorphic to the second cohomology group of G, $H_2(G, \mathbf{C}^*)$, in finite case, and also is isomorphic to the well-known notion the *Schur multiplier* of G, denoted by $M(G)$(see [14, 15].

If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, then the Baer-invariant of the group G will be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, \ _c F]} \ ,$$

where $\gamma_{c+1}(F)$ is the $(c+1)$st term of the lower central series of F and $[R,_1 F] = [R,F]$, $[R,_c F] = [[R,_{c-1} F], F]$, inductively. The above notion is also called the c-nilpotent multiplier of G and denoted by $M^{(c)}(G)$ (see [7]).

If $\mathcal{V} = \mathcal{N}_{c_1,\ldots,c_t}$, the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, then

$$\mathcal{N}_{c_1,\ldots,c_t} M(G) = \frac{R \cap \gamma_{c_t+1} \circ \cdots \circ \gamma_{c_1+1}(F)}{[R,_{c_1} F,_{c_2} \gamma_{c_1+1}(F), \cdots,_{c_t} \gamma_{c_{t-1}+1} \circ \cdots \circ \gamma_{c_1+1}(F)]} \ ,$$

where $\gamma_{c_t+1} \circ \cdots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\cdots(\gamma_{c_1+1}(F))\cdots))$ are the terms of iterated lower central series of F. See [4, corollary 6.14] for the following equality

$$N_{c_1,\ldots,c_t}(R,F) = [R,_{c_1} F,_{c_2} \gamma_{c_1+1}(F), \ldots,_{c_t} \gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)].$$

We shall also call $\mathcal{N}_{c_1,\dots,c_t}M(G)$, the $(c_1,\dots,c_t)$-polynilpotent multiplier of G.

Definition 1.1 [2]

The notion of *basic commutators* on letters $x_1, x_2, \dots, x_n, \dots$, are defined as follows:

(i) The letters $x_1, x_2, \dots, x_n, \dots$ are basic commutators of weight one, ordered by setting $x_i < x_j$ if $i < j$.

(ii) If basic commutators c_i of weight $wt(c_i) < k$ are defined and ordered, then define basic commutators of weight k by the following rules:

$[c_i, c_j]$ is a basic commutator of weight k if

1. $wt(c_i) + wt(c_j) = k$,
2. $c_i > c_j$,
3. if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.

Then continue the order by setting $c > c_i$ whenever $wt(c) > wt(c_i)$ and fixing any order amongst those of weight k and finally numbering them in order.

Theorem 1.2 (P.Hall [2, 3])

Let $F =< x_1, x_2, \dots, x_d >$ be a free group, then

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)} \quad , \qquad 1 \leq i \leq n$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \dots, n+i-1$ on the letters $\{x_1, \dots, x_d\}$.

Theorem 1.3 (Witt Formula [3])

The number of basic commutators of weight n on d generators is given by the following formula:

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m}$$

where $\mu(m)$ is the *Mobious function*, and defined to be

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = p_1^{\alpha_1} \dots p_k^{\alpha_k} \quad \exists \alpha_i > 1, \\ (-1)^s & \text{if } m = p_1 \dots p_s, \end{cases}$$

Our main results are as follows that they are the generalization of [9].

Theorem A. Let $\mathcal{N}_{c_1,c_2,\ldots,c_t}$ be the polynilpotent variety of class row $(c_1, c_2, \ldots, c_t)$ and G be a finitely generated abelian group. Then for all $n \geq 0$

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Ext^n_{\mathbf{Z}}(\mathbf{Z}_m, G)) \cong Ext^n_{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)).$$

If G is finite,

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Ext^n_{\mathbf{Z}}(G, \mathbf{Z}_m)) \cong Ext^n_{\mathbf{Z}}(\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G), \mathbf{Z}_m).$$

Theorem B. If G is a finite abelian group and $\mathcal{N}_{c_1,c_2,\ldots,c_t}$ be the polynilpotent variety of class row $(c_1, c_2, \ldots, c_t)$, then for all $n \geq 0$

$$(i)\ \mathcal{N}_{c_1,c_2,\ldots,c_t}M(Tor^{\mathbf{Z}}_n(\mathbf{Z}_m, G)) \cong Tor^{\mathbf{Z}}_n(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)).$$

$$(ii)\ \mathcal{N}_{c_1,c_2,\ldots,c_t}M(Tor^{\mathbf{Z}}_n(G, \mathbf{Z}_m)) \cong Tor^{\mathbf{Z}}_n(\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G), \mathbf{Z}_m).$$

2. Elementary Results

The following theorem permits us to look at the notion of the Baer-invariant as a functor.

Theorem 2.1

Let $\mathcal{V}$ be an arbitrary variety of groups. Then, using the notion of the Baer-invariant, we can consider the following covariant functor from the category of all groups, $\mathcal{G}roup$, to the category of all abelian groups, $\mathcal{A}b$

$$\mathcal{V}M(-)\ :\ \mathcal{G}roup \longrightarrow \mathcal{A}b\ ,$$

which assigns to any group G the abelian group $\mathcal{V}M(G)$.

Proof. See [7]. □

In [8] have been shown that the Baer-invariant functor is *not additive* nor *right and left* exact even if we restrict ourself to abelian groups.

In 1952, C. Miller [13] proved that the Schur multiplier of a free product is isomorphic to the direct sum of the Schur multipliers of the free factors. In other words, he proved that the Schur multiplier functor $M(-)$ *is* *coproduct-preserving.* But the second nilpotent multiplier functor, $\mathcal{N}_2M(-)$, *is not coproduct preserving*, in general [8]. Also, in 1980 M.R.R. Moghaddam [12] proved that in general, the Baer-invariant functor commutes with direct limit of a directed system of groups.

Theorem 2.2 (M.R.R. Moghaddam [12])

Let $\{G_i; \alpha_i^j, I\}$ be a direct system of groups and $\mathcal{V}$ an arbitrary variety of groups. Then

$$\mathcal{V}M(\overset{lim}{\rightarrow} G_i) \cong \overset{lim}{\rightarrow} \mathcal{V}M(G_i)\ .$$

In the rest, we need an explicit formula for the polynilpotent multiplier of a finitely generated abelian groups which is recently proved by B. Mashayekhy and M. Parvizi as follows.

Theorem 2.3 (B. Mashayekhy and M. Parvizi [11])

Let $\mathcal{N}_{c_1,c_2,\ldots,c_t}$ be the polynilpotent variety of class row $(c_1, c_2, \ldots, c_t)$ and $G \cong \mathbf{Z}^{(n)} \oplus \mathbf{Z}_{n_1} \oplus \ldots \oplus \mathbf{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq \imath \leq k-1$. Then an explicit structure of the polynilpotent multiplier of G is as follows

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G) \cong \mathbf{Z}^{(f_n)} \oplus \mathbf{Z}_{n_1}^{(f_{n+1}-f_n)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(f_{n+k}-f_{n+k-1})}\quad ,$$

where $f_i = \chi_{c_t+1}(\chi_{c_{t-1}+1}(\ldots(\chi_{c_1+1}(i))\ldots))$ for all $n \leq i \leq n+k$.

In Theorem 2.3 consider $t = 1$ and $n = 0$, then we immediately conclude the following theorem which it is obtained in 1997 as follows.

Theorem 2.4 (B. Mashayekhy and M.R.R. Moghaddam [10])

Let $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}$, be a finite abelian group, where $n_{i+1}|n_i$ for all $1 \leq i \leq k-1$ and $k \geq 2$. Then, for all $c \geq 1$, the c-nilpotent multiplier of G is

$$\mathcal{N}_cM(G) \cong \mathbf{Z}_{n_2}^{(\chi_{c_1+1}(2))} \oplus \mathbf{Z}_{n_3}^{(\chi_{c_1+1}(3)-\chi_{c_1+1}(2))} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(\chi_{c_1+1}(k)-\chi_{c_1+1}(k-1))}.$$

By using Theorem 2.4, one can see that the c-nilpotent multiplier functors can preserve every elementary abelian p-group.

3. Main Results

In this section, we will see the behaviour of the functor $\mathcal{N}(c_1, \ldots, c_t)M(-)$ with the functors $Ext^n_{\mathbf{Z}}(A, -)$, $Ext^n_{\mathbf{Z}}(-, A)$ and $Tor^{\mathbf{Z}}_n(A, -)$. We know for all $n \geq 2$, $Tor^{\mathbf{Z}}_n(A, B) = 0$ and $Ext^n_{\mathbf{Z}}(A, B) = 0$ (see [16]). So, we concentrate ourselves on $n = 0, 1$. First, we need the following lemmas.

Lemma 3.1

For any abelian groups A and B, we have

(i) $Ext^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, B) \cong B/mB$.

(ii) If A and B are finite abelian groups, then

$$Ext^1_{\mathbf{Z}}(A, B) \cong Ext^1_{\mathbf{Z}}(B, A).$$

(iii) $Tor^{\mathbf{Z}}_1(\mathbf{Z}/m\mathbf{Z}, B) \cong B[m]$, where $B[m] = \{b \in B : mb = 0\}$.

(iv) $Tor^{\mathbf{Z}}_1(A, B) \cong Tor^{\mathbf{Z}}_1(B, A)$.

Proof. See [16, Chapters 7, 8]. □

Lemma 3.2

Let A and $\{B_k\}_{k\in I}$ be abelian groups. Then for all $n \geq 0$ the following isomorphism hold.

(i) $Ext^n_{\mathbf{Z}}(A, \prod_{k\in I} B_k) \cong \prod_{k\in I} Ext^n_{\mathbf{Z}}(A, B_k)$, $Ext^n_{\mathbf{Z}}(\coprod_{k\in I} B_k, A) \cong \prod_{k\in I} Ext^n_{\mathbf{Z}}(B_k, A)$.

(ii) $Tor^{\mathbf{Z}}_n(A, \coprod_{k\in I} B_k) \cong \coprod_{k\in I} Tor^{\mathbf{Z}}_n(A, B_k)$, $Tor^{\mathbf{Z}}_n(\coprod_{k\in I} B_k, A) \cong \coprod_{k\in I} Tor^{\mathbf{Z}}_n(B_k, A)$.

Proof. See [16]. □

It is obvious that the functor $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(-)$ commutes with the functors $Ext^n_{\mathbf{Z}}(\mathbf{Z}_m, -)$, and $Tor^{\mathbf{Z}}_n(\mathbf{Z}_m, -)$ for all $n \geq 2$, by lemma 3.2. Now we are going to pay our attention to the functors of $Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, -)$, $Ext^1_{\mathbf{Z}}(-, \mathbf{Z}_m)$ and $Tor^{\mathbf{Z}}_1(\mathbf{Z}_m, -)$.

Theorem 3.3

Let $G \cong \mathbf{Z}^{(n)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}$, be a finitely generated abelian group, where $n \geq 0$, $n_{i+1}|n_i$ for all $1 \leq i \leq k-1$. Then, for all $(c_1, c_2, \ldots, c_t)$, the following isomorphisms hold.

(i) $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, G)) \cong \mathbf{Z}_m^{(f_n)} \oplus (\oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}^{(f_{n+i}-f_{n+i-1})})$.

(ii) $Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)) \cong \mathbf{Z}_m^{(f_n)} \oplus (\oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}^{(f_{n+i}-f_{n+i-1})})$.
(iii) $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Ext^1_{\mathbf{Z}}(G, \mathbf{Z}_m)) \cong \oplus \sum_{i=2}^{k} \mathbf{Z}_{(m,n_i)}^{(f_i-f_{i-1})}$.
(iv) $Ext^1_{\mathbf{Z}}(\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G), \mathbf{Z}_m) \cong \oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}^{(f_{n+i}-f_{n+i-1})}$.
(v) $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, G)) \cong \oplus \sum_{i=2}^{k} \mathbf{Z}_{(m,n_i)}^{(f_i-f_{i-1})}$.
(vi) $Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)) \cong \oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}^{(f_{n+i}-f_{n+i-1})}$.

Proof. (i) By Lemma 3.3(i), $Ext^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}/m\mathbf{Z} \cong \mathbf{Z}_m$. Now by using Lemmas 3.3(i) and 3.2(i), we have

$$Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, G) \cong (Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}))^{(n)} \oplus (\oplus \sum_{i=1}^{k} Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_{n_i}))$$

$$\cong \mathbf{Z}_m^{(n)} \oplus (\oplus \sum_{i=1}^{k} \mathbf{Z}_{n_i}/m\mathbf{Z}_{n_i}) \cong \mathbf{Z}_m^{(n)} \oplus (\oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}).$$

Now, by Theorem 2.2 and noting that $(m, n_{i+1})|(m, n_i)|m$ we have

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, G))$$

$$\cong \mathbf{Z}_m^{(f_2-f_1)} \oplus \mathbf{Z}_m^{(f_3-f_2)} \oplus \ldots \oplus \mathbf{Z}_m^{(f_n-f_{n-1})} \oplus \mathbf{Z}_{(m,n_1)}^{(f_{n+1}-f_n)} \oplus \ldots \oplus \mathbf{Z}_{(m,n_k)}^{(f_{n+k}-f_{n+k-1})}$$

$$\cong \mathbf{Z}_m^{(f_n)} \oplus (\oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}^{(f_{n+i}-f_{n+i-1})}).$$

(ii) By Theorem 3.1 and Lemmas 3.3(i) and 3.2(i), we have

$$Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G))$$

$$\cong (Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}))^{(f_n)} \oplus (\oplus \sum_{i=1}^{k} (Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_{n_i}))^{(f_{n+i}-f_{n+i-1})})$$

$$\cong \mathbf{Z}_m^{(f_n)} \oplus (\oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}^{(f_{n+i}-f_{n+i-1})}).$$

(iii) and (iv) By similar methods of (i) and (ii).

(v) By Lemmas 3.3(ii) and 3.2(ii) we have

$$Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, G) \cong (Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}))^{(n)} \oplus (\oplus \sum_{i=1}^{k} Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_{n_i})) \cong \oplus \sum_{i=1}^{k} \mathbf{Z}_{n_i}[m].$$

Note that $Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}) \cong 1$ and $\mathbf{Z}_n[m] \cong \mathbf{Z}_{(m,n)}$. So we have $Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, G) \cong \oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}$. Now by Theorem 2.2 the result holds.

(vi) Again by using Theorem 3.1 and Lemmas 3.3(ii) and 3.2(ii), we have

$$Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)) \cong (Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}))^{(f_n)} \oplus (\oplus \sum_{i=1}^{k} Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_{n_i}^{(f_{n+i}-f_{n+i-1})})$$

$$\cong \oplus \sum_{i=1}^{k} \mathbf{Z}_{(m,n_i)}^{(f_{n+i}-f_{n+i-1})}. \ \square$$

This theorem means that the polynilpotent multiplier functors of class row $c_1, c_2, \ldots, c_t$, $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(-)$ *do not* commute with $Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, -)$ and $Ext^1_{\mathbf{Z}}(-, \mathbf{Z}_m)$ in infinite case.

Now you can find the proof of main results of the paper.

Proof A. (i) It is clear by parts (i), (ii) of the previous theorem.
(ii) The parts (iii) and (iv) of the previous theorem conclude the result. $\square$

Proof B. Since G is infinite, so $n \geq 1$. Hence the result holds by the previous theorem parts (v) and (vi). The second part is similar to first one. $\square$

We know that $Hom(\mathbf{Z}_m, \mathbf{Z}) \cong 1$ and $Hom(\mathbf{Z}, \mathbf{Z}_m) \cong \mathbf{Z}_m$. So by similar methods of Theorem 3.4 we are going to look at the behaviour of functor $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(-)$ with $Ext^0_{\mathbf{Z}}(\mathbf{Z}_m, -) = Hom(\mathbf{Z}_m, -)$, $Ext^0_{\mathbf{Z}}(-, \mathbf{Z}_m) = Hom(-, \mathbf{Z}_m)$ and $Tor_0^{\mathbf{Z}}(\mathbf{Z}_m, -) = \mathbf{Z}_m \otimes -$ as the following theorem.

Theorem 3.5

For any finitely generated abelian group $G \cong \mathbf{Z}^{(n)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}$, we have

(i) $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Hom(\mathbf{Z}_m, G) \cong \mathbf{Z}_{(m,n_2)}^{(f_2)} \oplus \ldots \oplus \mathbf{Z}_{(m,n_k)}^{(f_k-f_{k-1})}$.

(ii) $Hom(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)) \cong \mathbf{Z}_{(m,n_1)}^{(f_{n+1}-f_n)} \oplus \ldots \oplus \mathbf{Z}_{(m,n_k)}^{(f_{n+k}-f_{n+k-1})}$.

(iii)If G is finite, then $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Hom(\mathbf{Z}_m, G) \cong Hom(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G))$.
If G is infinite, then $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Hom(\mathbf{Z}_m, G) \ncong Hom(\mathbf{Z}_m, \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G))$.

(iv)

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Hom(G, \mathbf{Z}_m)) \cong Hom(\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G), \mathbf{Z}_m)$$

$$\cong \mathbf{Z}_m^{(f_n)} \oplus \mathbf{Z}_{(m,n_1)}^{(f_{n+1}-f_n)} \oplus \ldots \oplus \mathbf{Z}_{(m,n_k)}^{(f_{n+k}-f_{n+k-1})}.$$

(v)

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(\mathbf{Z}_m \otimes G) \cong \mathbf{Z}_m \otimes \mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)$$

$$\cong \mathbf{Z}_m^{(f_n)} \oplus \mathbf{Z}_{(m,n_1)}^{(f_{n+1}-f_n)} \oplus \ldots \oplus \mathbf{Z}_{(m,n_k)}^{(f_{n+k}-f_{n+k-1})}.$$

Now, in the following we are going to show that our conditions in the previous results are essential. In general case $Ext^i_{\mathbf{Z}}(A,-)$, $Ext^i_{\mathbf{Z}}(-,A)$ and $Tor^{\mathbf{Z}}_i(A,-)$, where A is not cyclic, *do not commute* with $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(-)$, for $i=0,1$.

Example (i)**.** Let $A \cong \mathbf{Z}_r \oplus \mathbf{Z}_s$ and $G \cong \mathbf{Z}_m \oplus \mathbf{Z}_n$, where $n|m$ and $m|(r,s)$. Then

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(Ext^1_{\mathbf{Z}}(A,G) \cong \mathbf{Z}_m^{(f_2)} \oplus \mathbf{Z}_n^{(f_4-f_2)} \ncong \mathbf{Z}_n^{(2f_2)} \cong Ext^1_{\mathbf{Z}}(A,\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G))$$

i.e the functor $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(-)$ does not commute with $Ext^1_{\mathbf{Z}}(A,-)$. Similarly the polynilpotent functor does not commute with the following functors: $Ext^i_{\mathbf{Z}}(A,-),\ Ext^i_{\mathbf{Z}}(-,A),\ Tor^{\mathbf{Z}}_i(A,-)$ for $i=0,1$.

Example (ii)**.** $Hom(S_n,\mathbf{Z}_2) \cong \mathbf{Z}_2$, for $n \geq 2$. Also we know that $M(S_n) \cong \mathbf{Z}_2$, for each $n \geq 4$, see [6, theorem 2.12.3]. Thus

$$M(Hom(\mathbf{Z}_2,S_n)) \cong M(Hom(S_n,\mathbf{Z}_2)) \cong 1$$

$$\ncong \mathbf{Z}_2 \cong Hom(M(S_n),\mathbf{Z}_2) \cong Hom(\mathbf{Z}_2,M(S_n)).$$

$$M(S_n \otimes \mathbf{Z}_2)) \cong 1 \ncong \mathbf{Z}_2 \cong M(S_n) \otimes \mathbf{Z}_2.$$

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