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On Topological Extensions with Continuous Sections

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Abstract

Let Q be a topological group , K a trivial Q-module and $e : 0 \to K \to G \to Q \to 0$ a topological extension. It is known that an extension may not have a continuous section i.e. $u : Q \to G$ such that $\pi u = Id_Q$. In this paper we show that if K is locally compact metrizable and Q locally compact , totally disconnected ,metrizable then the extension (e) has a continuous section.

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Introduction

Let Q be a topological group K a trivial Q-module . A *topological extension* of Q by K is a short exact sequence $0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$,where i is a topological embedding onto a closed subgroup and π an open continuous onto homomorphism. The continuous map $u : Q \to G$ is a *section* if $\pi u = Id_Q$.
The class of extensions with continuous sections can be characterized by the second cohomology [3]. But there are extensions with out continuous sections (see section 1).
The main question we consider is as follows: when an extension has a continuous section?. In this paper we show that if K is locally compact ,metrizable and Q locally compact , totally disconnected ,metrizable then the corresponding extension has a continuous section.
In section 1, we give some examples of extensions without sections. In section 2 , the main result is proved. Notation and definitions as in [4].

1 Topological extensions

The importance of extensions with continuous sections is based on the fact that ,by using the methods of [2], there is an isomorphism between the second cohomology and the group of extensions with sections [3], namely

$$H^2(Q,A) = Ext_s(Q,A)$$

Some extensions may have continuous sections. For example ,if Q is a connected locally compact group ,then any topological extension of Q by a connected simply connected Lie group has a continuous section [5,theorem 2].
Remark . If $0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$ has a continuous section then G and $K \times Q$ are homeomorphic [1] .

There are extensions that do not admit continuous sections .

Example 1.1. Let $Q = SO_3(R)$, and G be it,s two sheeted covering space $SU_2(C)$, so we set $A = Z_2$. Certainly G is an extension of Q by A. If the extension has a continuous section then by the above remark G and $A \times Q$ are homeomorphic. But there is no such map from $Q = SO_3(R)$ to $SU_2(C)$ since $SU_2(C)$ is connected while $Z_2 \times SO_3(R)$ is not .

Example 1.2. Let $Q = S^1$, $G = S^1$ (the circle) and $\pi : G \to Q$ be the two fold covering map . Then $(e) : 0 \to Z_2 \xrightarrow{i} S^1 \xrightarrow{\pi} S^1 \to 0$ is an extension . If (e) has a continuous section then by the above remark $S^1 \simeq S^1 \times Z_2$. This is a contradiction since S^1 is connected but $S^1 \times Z_2$ is not.
Example 1.3 . Let S^1 be a circle and $\sigma : S^1 \to S^1$, $\sigma(z) = z^2$, $z \in S^1$. By example 1.2 $(e) : 0 \to Z_2 \xrightarrow{i} S^1 \xrightarrow{\sigma} S^1 \to 0$ has no continuous section. Let $\pi_1 : S^1 \times (Z_2)^\infty \to S^1$ be the projection onto the first factor. Then the kernel of $\pi_1\sigma : S^1 \times (Z_2)^\infty \to S^1$ is $Z_2 \times (Z_2)^\infty$ which is isomorphic to $(Z_2)^\infty$. Now

$0 \to Z_2 \times (Z_2)^\infty \to S^1 \xrightarrow{\pi_1\sigma} S^1 \to 0$ is a topological extension.Consider the following diagram :

$$(*) \quad \begin{tikzcd}
0 \arrow[r] & Z_2 \times (Z_2)^\infty \arrow[r] & S^1 \times (Z_2)^\infty \arrow[r, "\sigma\pi_1"] \arrow[d, "\pi_1"'] & S^1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z_2 \arrow[r] & S^1 \arrow[r, "\sigma"] & S^1 \arrow[r] & 0
\end{tikzcd}$$

The right square of the diagram commutes and the top row has no continuous section ; for if it has a section then the bottom row has one ,but this is a contradiction by example 1.2 .

2 Extensions with continuous sections

In this section we show that under certain conditions a map may have a continuous section. Recall that a topological space is *zero-dimensional* if for every open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X there is a pairwise disjoint open cover $\{\nu_i\}$ which refines $\{U_\alpha\}_{\alpha\in\Lambda}$. A topological space is *totally disconnected* if all of its components are points [4].

Proposition 2.1 *Let X be a metric space ,Y a zero-dimensional metric space . Suppose $f : X \to Y$ is an open map such that $f^{-1}(y)$ is complete in X for each $y \in Y$. Then there is a sequence $\{L_i\}$ of open subsets of X and $\{\nu_i\}$ of open covers of Y such that*

(1) $\overline{L_{i+1}} \subseteq L_i$

(2) $f(L_i) = Y$

(3) $diam(f^{-1}(v) \cap L_i) < 1/i$ *for all* $v \in \nu_i$

Proof.(By induction). Let $L_0 = X$. Let $i \geq 1$ and inductively assume L_{i-1} is an open subset of X such that $f(L_{i-1}) = Y$. By induction there is a collection ν of open subsets of X such that ν covers L_{i-1} and $\overline{U} \subset L_{i-1}$ and $\mathrm{diam}U < 1/i$ for each $U \in \nu$. Since $f : X \to Y$ is open and $f(L_{i-1}) = Y$ then $f\nu = \{f(U); U \in \nu\}$ is an open cover of Y. Since Y is zero-dimensional there is a pairwise disjoint open cover ν_i of Y which refines $f\nu$. For each $v \in \nu_i$, choose $U_v \in \nu$ so that $v \subset f(U_v)$. Set $L_i = \cup\{f^{-1}(v)\cap U_v; v \in \nu_i\}$. L_i is clearly open in X. Now we show that $L_i \subset L_{i-1}$. Let $K = \cup\{f^{-1}(v) \cap \overline{U_v}; v \in \nu_i\}$. Clearly $L_i \subset K \subset L_{i-1}$. If $x \in X - K$ and $v \in \nu_i$ so that $f(x) \in v$, then $f^{-1}(v) \cap (X - \overline{U_v})$ is an open neighborhood of x in X which is disjoint from K. So $X - K$ is open which implies K is closed. Hence $\overline{L_i} \subset K \subset L_{i-1}$.
Now let $y \in Y$. Choose $v \in \nu_i$ so that $y \in v$. Since $v \subset f(U_\nu)$,there is an $x \in U_v$ such that $f(x) = y$. This shows that $f(L_i) = Y$.
Since distinct elements of ν_i are disjoint so are distinct elements of $f^{-1}(\nu_i) = \{f^{-1}(v); v \in \nu_i\}$. Thus for each $v \in \nu_i$, $f^{-1}(v) \cap L_i = f^{-1}(v) \cap U_v$. So

$$diam(f^{-1}(v) \cap L) < diamU_v < 1/i \quad foreach\ v \in \nu_i$$

Theorem 2.2 *Let X be a metric space ,Y a zero-dimensional metric space . Suppose $f : X \to Y$ be an open map such that $f^{-1}(y)$ is complete in X for each $y \in Y$. Then there exists a map $g : Y \to X$ such that $f \circ g = 1_Y$.*

Proof. Suppose $\{L_i\}$ is the sequence as in proposition 2.1. We define a map $g : Y \to X$ as follows. Let $M = \cap_{i=1}^{\infty}\overline{L_i}$. For every $y \in Y$, $\{f^{-1}(y) \cap \overline{L_i}\}$ is a decreasing sequence of non-empty closed subsets of $f^{-1}(y)$ with the diameter

converging to 0 . Since the metric on $f^{-1}(y)$ is complete , then $f^{-1}(y) \cap M = \cap_{i=1}^{\infty}(f^{-1}(y) \cap \overline{L_i})$ is a singleton.
Define $g : Y \to X$ by $g(y) = f^{-1}(y) \cap M$ for each $y \in Y$. Clearly $f \circ f(y) = y$,$y \in Y$.
Now we show that g is continuous. Let $y \in Y$ and $\epsilon > 0$. Choose $i \geq 0$ so that $1/i < \epsilon$ and choose $v \in \nu_i$ so that $y \in v$. Since $g(v) \subset f^{-1}(y) \cap M \subset f^{-1}(y) \cap L_i$, then diam$g(v) < \epsilon$.

Definition 2.3 Let G be a topological group. A sequence $\{x_i\}$ in G is a *left (right) Cauchy* if for every neighborhood U of identity there is a $k \geq 1$ such that $i, j \geq k$ implies $x_i^{-1}x_j \in U$ $(x_i x_j^{-1} \in U)$. G is *left (right) complete* if and only if every left (right) Cauchy sequence in G converges.

In the following results "left" can be replaced by "right".

Proposition 2.4 . *Let G be a metrizable topological group. The following are equivalent:*

(i) G is left complete

(ii) Some left invariant metric on G is complete

(iii) Every left invariant metric on G is complete

Proof.(i)⇒ (iii). Let G be a left complete space. Assume ρ is a left invariant metric on G and $\{x_i\}$ a Cauchy sequence in G under ρ. Let $\epsilon > 0$. There is a $k \geq 0$ such that $i, j \geq k$ implies $\rho(x_i, x_j) < \epsilon$. Hence for $i, j \geq k$, $\rho(1, x_i^{-1}x_j) < \epsilon$. So $x_i^{-1}x_j$ is in the ϵ-neighborhood of identity. Hence $\{x_i\}$ is a left Cauchy sequence in G . So $\{x_i\}$ converges.
(iii)⇒(ii). It is obvious since every metrizable topological group has a left invariant metric [4]
(ii)⇒ (i). Suppose G has a complete left invariant metric ρ and $\{x_i\}$ a left Cauchy sequence in G . Let $\epsilon > 0$. Then there is a $k \geq 1$ such that $i, j \geq k$ implies $\rho(1, x_i^{-1}x_j) < \epsilon$. So $\rho(x_i, x_j) < \epsilon$. Hence $\{x_i\}$ is a Cauchy sequence under ρ. Consequently $\{x_i\}$ converges .Hence G is left complete.

Corollary 2.5 *If G is a locally compact metrizable topological group then every left invariant metric on G is complete.*

Proof. Let $\{x_i\}$ be a left Cauchy sequence in G. Let N be a compact neighborhood of 1 in G . Then for $k \geq 1$, $i \geq k$ implies $x_k^{-1}x_i \in N$. Then $x_i \in x_k N$ for $i \geq k$. Since $x_k N$ is compact it follows that $\{x_i\}$ has a convergent subsequence $\{y_i\}$. Let $y_i \to z$, $z \in G$. We show that $x_i \to z$. Let U be a neighborhood of z . There is a neighborhood V of identity in G such that

$zVV \subset U$ [4]. There is $k \geq 1$ such that $i, j \geq k$ implies $x^{-1}x_j \in V$ and $y_k \in zV$. Since $y_k = x_j$ implies $j \geq k$,then $j \geq k$ implies $y_k^{-1}x_j \in V$. Hence for $j \geq k$, $x_j \in y_kV \subset zVV \subset U$. Therefore , G is left complete . Now the result follows by proposition 2.4.

Corollary 2.6 *Let K be a closed subgroup of a metrizable topological group G . If G/K is zero-dimensional and K has a complete left-invariant metric , then G is homeomorphic to $K \times G/K$.*

Proof. Let $\pi : G \to G/K$ be the natural quotient map $\pi(g) = gK$, $g \in G$. Since G/K has the quotient topology , π is open and G/K is metrizable.

Let ρ be a left-invariant metric on G. Since K has a complete left-invariant metric then it follows by proposition 2.4 that ρ restricts to a complete metric on K. Since ρ is left-invariant , then ρ restricts to a complete metric on gK for each $g \in G$.Hence for any $y \in G/K$, π^{-1} is complete under ρ. Now by theorem 2.2 there is a map $\sigma : G/K \to G$such that $\pi\sigma = 1_G$.We define $\phi : G \to G{\times}G/K$ by $\phi(g) = (\sigma\pi(g))^{-1}g, \pi(g))$, $\psi : K \times G/K \to G, \psi(k, y) = \sigma(y)K, (k, y) \in K \times G/K$. We show that $\phi(G) \subset K \times G/K$. It is enough to show that $(\sigma\pi(g))^{-1}g \in K$. Let $g \in G$. Since $\pi\sigma\pi(g) = \pi(g)$, then $\pi\sigma\pi(g) \in= \pi(g)$ so $\sigma\pi(g) = gk$ for some $k \in K$. Hence $\sigma\pi(g) \in gK$.

and $g^{-1}(\sigma\pi(g)) \in K$. So $(\sigma\pi(g)))^{-1}g = (g^{-1}(\sigma\pi(g)))^{-1} \in K$.
Next we show that $\psi\phi$ are inverse of each other. Let $g \in G$. Then $\psi\pi(g) = \psi((\sigma\pi(g)))^{-1}g, \pi(g)) = \sigma(\pi(g))((((\sigma \circ \pi(g)))^{-1}g = g$.

Let $(k, y) \in K{\times}G/K$. $\phi\psi(k, y) = \phi(\sigma(y)k) = ((\sigma{\circ}\pi(\sigma(y)k))^{-1}\sigma(y)k, \pi((\sigma(y)k))$
Since $\sigma(y)k) \in \sigma(y)K = \pi^{-1}(\pi \circ \sigma(y))$, then $\pi(\sigma(y)k) = \pi \circ \sigma(y) = y$. Thus $\phi\psi(k, y) = ((\sigma(y))^{-1}\sigma(y)k, y) = (k, y)$
Therefore, $\phi : G \to K \times G/K$ is a homeomorphism.

Proposition 2.7 *Let $0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$ be a topological extension of K by Q. Suppose K is locally compact and Q locally compact ,totally disconnected metrizable topological group and G a metric space. Then there exists a continuous map $u : Q \to G$ such that $\pi u = 1_Q$.*

Proof. By [4], Q is zero-dimensional topological group.For any $y \in Q$, $\pi^{-1}(y)$ is isomorphic to K. Hence $\pi^{-1}(y)$ is closed Since K and Q are locally compact , then G is locally compact[4]. By corollary 2.5, G is a complete space. So , as a closed subset of G, $\pi^{-1}(y)$ is complete. Now by theorem 2.2 there exists a continuous map $u : Q \to G$ such that $\pi u = 1_Q$.

Remark 2.8 . In Proposition 2.7 if we replace "metrizable" by "first countable" then by [4] G and K are metrizable. Hence we will have a continuous section.

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