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On I-Lacunary Strong Convergence in $2-$Normed Spaces

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Abstract. In this article we introduce I- lacunary convergence of sequences in 2-normed spaces.

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1. Introduction

Many author studied various sequence spaces using normed or seminormed linear spaces.

In this article, using lacunary sequences and the notion of ideal, we aimed to introduce some new sequence spaces with respect to a modulus function in $2-$normed linear spaces. By an ideal we mean a family $\mathcal{I} \subset 2^Y$ of subsets a nonempty set Y satisfying: (**i**) $\emptyset \in \mathcal{I}$; (**ii**) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; (**iii**) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ [10, 11]. By lacunary sequence we mean an increasing sequence $\theta = \{k_r\}$ of positive integers satisfying; $k_0 = 0$ and $h_r := k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. We denote the intervals, which θ determines, by $I_r := (k_{r-1}, k_r]$. The space of lacunary strongly convergent sequences was defined by Freedman et al.[14]

The notion of ideal convergence was introduced first by P. Kostyrko et al [10] as a generalization of statistical convergence.

The concept of $2-$normed spaces was initially introduced by Gähler [5] in the 1960's. Since then, this concept has been studied by many authors, see for instance [6].

Sahiner et al., introduce $\mathcal{I}-$convergence in $2-$normed spaces [8].

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}-$convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [10, 11].

A $2-$norm on a real vector space X of dimension $2 \leq d < \infty$ is a function $\|.,.\| : X \times X \to R$ which satisfies (**i**) $\|x,y\| = 0$ if and only if x and y are linearly dependent; (**ii**) $\|x,y\| = \|y,x\|$; (**iii**) $\|\alpha x, y\| = |\alpha| \|x,y\|$, $\alpha \in \mathbb{R}$; (**iv**) $\|x, y+z\| \leq \|x,y\| + \|x,z\|$. The pair $(X, \|.,.\|)$ is then called a $2-$normed space [6]. As an example of a $2-$normed space we may take $X = \mathbb{R}^2$ being equipped with standard and Euclid $2-$norms on $\mathbb{R}^2$ are given by

$$\|x_1, x_2\|_S = \begin{vmatrix} < x_1, x_1 > & < x_1, x_2 > \\ < x_2, x_1 > & < x_2, x_2 > \end{vmatrix}^{\frac{1}{2}}$$

and

$$\|x_1, x_2\|_E = abs\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right), \qquad x_1 = (x_{11}, x_{12}) \text{ and } x_2 = (x_{21}, x_{22})$$

respectively, where $< .,. >$ stands for the inner product on X. We know that $(X, \|.,.\|)$ is a $2-$Banach space if every Cauchy sequence in X is convergent to some x in X.

Recall that a modulus function is a function $f : [0,\infty) \to [0,\infty)$ such that
(i) $f(x) = 0$ if and only if $x = 0$;
(ii) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$;
(iii) f is increasing;
(iv) f is continuous from the right at zero.

2. DEFINITIONS AND INCLUSION THEOREMS

Let I be an admissible ideal, f be a modulus function, $(X, \|.,.\|)$ be a $2-$normed space and $p = p_k$ be a sequence of positive real numbers. By $S(2-X)$ we denote the space of all sequences defined over $(X, \|.,.\|)$.

Definition 1.

$$[N_\theta, f, p, \|,.,\|]^I = \left\{ \begin{array}{c} x \in S(2-X) : \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} [f(\|x_k - L, z\|)]^{p_k} \geq \varepsilon \right\} \in I \\ \textit{for some } L > 0 \textit{ and each } z \in X \end{array} \right\}.$$

$$[N_\theta, f, p, \|,.,\|]_0^I = \left\{ \begin{array}{c} x \in S(2-X) : \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} [f\|x_k, z\|]^{p_k} \geq \varepsilon \right\} \in I \\ \textit{for each } z \in X \end{array} \right\}.$$

The following well-known inequality will be used in the study. If

$$0 \leq p_k \leq \sup p_k = H, \; D = \max\left(1, 2^{H-1}\right)$$

then

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

for all k *and* $a_k, b_k \in \mathbb{C}$.

Theorem 1. *If* (p_k) *is bounded then* $[N_\theta, f, p, \|, ., \||]^I$, $[N_\theta, f, p, \|, ., \||]_0^I$, $[N_\theta, f, p, \|, ., \||]_\infty^I$ *are linear spaces. We will prove the assertion for* $[N_\theta, f, p, \|, ., \||]_0^I$; *the others can be proved similarly.*

Proof. Assume that $x, y \in [N_\theta, f, p, \|, ., \||]_0^I$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. So

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} [f \, \|x_k, z\||]^{p_k} \ge \varepsilon \right\} \in I$$

and

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} [f \, \|y_k, z\||]^{p_k} \ge \varepsilon \right\} \in I.$$

Since $\|, ., \|$ is a $2-$norm and f is an modulus function the following inequality holds:

$$\begin{aligned} h_r^{-1} \sum_{k \in I_r} [f(\|(\alpha x_k + \beta y_k), z\|)]^{p_k} &\le D h_r^{-1} T_{\alpha_1}^{\sup p_k} \sum_{k \in I_r} [f(\|x_k, z\|)]^{p_k} \\ &+ D h_r^{-1} T_{\alpha_2}^{\sup p_k} \sum_{k \in I_r} [f(\|y_k, z\|)]^{p_k}, \end{aligned}$$

where T_{α_1} and T_{α_2} are positive integers such that $|\alpha_1| \le T_{\alpha_1}$ and $|\alpha_2| \le T_{\alpha_2}$. On the other hand from the above inequality we get

$$\begin{aligned} &\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} [f(\|(\alpha x_k + \beta y_k), z\|)]^{p_k} \ge \varepsilon \right\} \\ \subseteq &\left\{ r \in \mathbb{N} : D h_r^{-1} T_{\alpha_1}^{\sup p_k} \sum_{k \in I_r} [f(\|x_k, z\|)]^{p_k} \ge \varepsilon \right\} \\ &\cup \left\{ r \in \mathbb{N} : D h_r^{-1} T_{\alpha_2}^{\sup p_k} \sum_{k \in I_r} [f(\|y_k, z\|)]^{p_k} \ge \varepsilon \right\} \end{aligned}$$

Two sets on the right hand side belongs to I and this completes the proof. ∎

Lemma 2. *Let* f *be a modulus function and let* $0 < \delta < 1$. *Then for each* $x > \delta$ *we have* $f(x) \le 2f(1)\delta^{-1}x$ [1].

Theorem 3. *Let* f *be a modulus function. Then* $[N_\theta, p, \|, ., \||]^I \subset [N_\theta, f, p, \|, ., \||]^I$

Proof. If $x \in [N_\theta, p, \|, ., \||]$ then for some $L > 0$ and each $z \in X$

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} [f(\|x_k - L, z\|)]^{p_k} \ge \varepsilon \right\} \in I.$$

Now let $\varepsilon > 0$ be given. We can choose $0 < \delta < 1$ such that for every t with $0 \le t \le \delta$ we have $f(t) < \varepsilon$. Now using the previous lemma we get

$$\begin{aligned} & \left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r,} \left[f\left(\|x_k, z\| \right) \right]^{p_k} \ge \varepsilon \right\} \\ = \; & \left\{ r \in \mathbb{N} : h_r^{-1} \left(h_r \max \left\{ \varepsilon^{\inf p_k}, \varepsilon^{\sup_k p_k} \right\} \right) \ge \varepsilon \right\} \\ & \cup \left\{ r \in \mathbb{N} : h_r^{-1} \max \left\{ \left(2f(1)\,\delta^{-1} \right)^{\inf p_k}, \left(2f(1)\,\delta^{-1} \right)^{\sup_k p_k} \right\} \sum_{k \in I_r} \left[\left(\|x_k, z\| \right) \right]^{p_k} \ge \varepsilon \right\} \end{aligned}$$

and this completes the proof. ∎

Theorem 4. *Let f be a modulus function. If $\lim_{t\to\infty} \sup \frac{f(t)}{t} = A > 0$ then $W^I(N_\theta, f, p, \|,.,\|) = W^I(N_\theta, p, \|,.,\|)$.*

Proof. It is sufficient only to show that $W^I(N_\theta, f, p, \|,.,\|) \subset W^I(N_\theta, p, \|,.,\|)$. If We have $\lim_{t\to\infty} \sup \frac{f(t)}{t} = A > 0$ then there exists a constant $B > 1\!\!\!/$ such $f(t) \ge Bt$ for all $t \ge 0$. Hence,

$$h_r^{-1} \sum_{k \in I_r} \left[f\left(\|x_k - L, z\| \right) \right]^{p_k} \ge h_r^{-1} B^{\sup p_k} \sum_{k \in I_r} \|x_k - L, z\|^{p_k}$$

and this inequality gives the result. ∎

More generally we have the following.

Theorem 5. *Let f_1 and f_2 be modulus functions. If $\lim_{t\to\infty} \sup \frac{f_1(t)}{f_2(t)} = A > 0$ then $[N_\theta, f_1(t), p, \|,.,\|]^I \subseteq [N_\theta, f_2(t), p, \|,.,\|]^I$.*

Proof. We know that if $\lim_{t\to\infty} \sup \frac{f_1(t)}{f_2(t)} = A > 0$ then there exists a constant $B > 1\!\!\!/$ such $f_1(t) \ge f_2(t)\, A$ for all $t \ge 0$. Hence,

$$h_r^{-1} \sum_{k \in I_r} \left[f_1\left(\|x_k - L, z\| \right) \right]^{p_k} \ge h_r^{-1} B^{\sup p_k} h_r^{-1} \sum_{k \in I_r} \left[f_2\left(\|x_k - L, z\| \right) \right]^{p_k}.$$

∎

Theorem 6. *Let $(X, \|.,.\|_S)$ and $(X, \|.,.\|_E)$ be standard and Euclid two normed spaces respectively then*

$$[N_\theta, f, p, \|,.,\|]^I \cap [N_\theta, f, p, \|,.,\|]^I \subseteq W^I\left(N_\theta, f, p, \left(\|,.,\|_E + \|,.,\|_S \right) \right).$$

Proof. We have the following inclusion.

$$\left\{r\in\mathbb{N}: h_r^{-1}\sum_{k\in I_r}\left[f\left(\left(\|,.,\|_E+\|,.,\|_S\right)(x_k-L,z)\right)\right]^{p_k}\geq\varepsilon\right\}$$

$$\subset\left\{r\in\mathbb{N}: Dh_r^{-1}\sum_{k\in I_r}\left[f\left(\|x_k-L,z\|_E\right)\right]^{p_k}\geq\varepsilon\right\}$$

$$\cup\left\{r\in\mathbb{N}: Dh_r^{-1}\sum_{k\in I_r}\left[f\left(\|x_k-L,z\|_S\right)\right]^{p_k}\geq\varepsilon\right\}.$$

∎

Theorem 7. *Let* $f,\ f_1,\ f_2,$ *be modulus functions. Then*

(*i*) $[N_\theta, f, p, \|,.,\|]_0^I \subseteq [\ f\circ f_1,\ ,\ p,\ \|,.,\|]_0^I$
(*ii*) $[N_\theta, f_1,\ p,\ \|,.,\|]_0^I \cap [N_\theta, f, p, \|,.,\|]_0^I \subseteq [f+f_1,\ p,\ \|,.,\|]_0^I$

Proof. For given $\varepsilon>0$, choose $0<\delta<1$ such that $0<t<\delta\ \Rightarrow f(t)<\varepsilon$.Let $(x_k)\ \in[N_\theta, f, p, \|,.,\|]$. On the other hand, we have

$$h_r^{-1}\sum_{k\in I_1}\left[f\left(f_1\|x_k,z\|\right)\right]^{p_k}+\sum_{k\in I_2}\left[f\left(f_1\|x_k,z\|\right)\right]^{p_k}$$

$$\leq\ \frac{1}{n}n\sup_k\varepsilon^{p_k}+\max\left(1,\left(K\delta^{-1}f(2)\right)^H\right)h_r^{-1}\sum_{k\in I_r}\left[f_1\left(\|x_k,z\|\right)\right]^{p_k}$$

Where the first summation is over $[f_1(\|x_k,z\|)]^{p_k}\leq\delta$ and the second one is over $[f_1(\|x_k,z\|)]^{p_k}>\delta$ and $K\geq 1$. ∎

(*ii*) Let $(x_k)\in[N_\theta, f_1,\ p,\ \|,.,\|]_0^I\cap[N_\theta, f, p, \|,.,\|]_0^I$. The fact

$$h_r^{-1}\left[f_1+f_2\left(\|x_k,z\|\right)\right]^{p_k}\leq Dh_r^{-1}\left\{\left[f_1\left(\|x_k,z\|\right)\right]^{p_k}+h_r^{-1}\left[f_2\left(\|x_k,z\|\right)\right]^{p_k}\right\}$$

gives us the result.

The following corollary can be obtained very similar to that of corresponding corollary of Tripathy at al [4].

Corollary 8. *Let* $f, f_1,\ f_2$ *be modulus functions. Then*

(*i*) $[N_\theta, f_1, p, \|,.,\|]_0^I \subseteq [N_\theta, f\circ f_1, p, \|,.,\|]_0^I,$
(*ii*) $[N_\theta, f_1, p, \|,.,\|]_0^I \cap [N_\theta, f_2, p, \|,.,\|]_0^I \subseteq [N_\theta, f_1+f_2, p, \|,.,\|]_0^I,$
(*iii*) $[N_\theta, f_1, p, \|,.,\|]_\infty^I \subseteq [N_\theta, f\circ f_1, p, \|,.,\|]_\infty^I,$
(*iv*) $[N_\theta, f_1, p, \|,.,\|]_\infty^I \cap\ [N_\theta, f_2, p, \|,.,\|] \subset [N_\theta, f_1+f_2, p, \|,.,\|]_\infty^I,$
(*v*) $[N_\theta, f_1, p, \|,.,\|]^I \subseteq [N_\theta, f\circ f_1, p, \|,.,\|]^I$
(*vi*) $[N_\theta, f_1, p, \|,.,\|]_0^I \cap [N_\theta, f_2, p, \|,.,\|]_0^I \subseteq [N_\theta, f_1+f_2, p, \|,.,\|]_0^I.$

3. SPACES DEFINED BY SEQUENCES OF MODULUS FUNCTIONS

Definition 2. *Let S be the space of sequences of modulus functions $F=(f_k)$ such that $\lim_{u\to 0^+}\sup_k f_k(u)=0$ and $(X,\|.,.\|)$ is a $2-$Banach space. We define the following spaces:*

$$[N_\theta,F,p,\|,.,\|]^I = \left\{ \begin{array}{c} x\in S(2-X):\left\{r\in\mathbb{N}:h_r^{-1}\sum_{k\in I_r}[f_k(\|x_k-L,z\|)]^{p_k}\geq\varepsilon\right\}\in I \\ \text{for some } L>0 \text{ and each } z\in X \end{array} \right\}$$

$$[N_\theta,F,p,\|,.,\|]_0^I = \left\{ \begin{array}{c} x\in S(2-X):\left\{r\in\mathbb{N}:h_r^{-1}\sum_{k\in I_r}[f_k(\|x_k,z\|)]^{p_k}\geq\varepsilon\right\}\in I \\ \text{for some } L>0 \text{ and each } z\in X \end{array} \right\}.$$

Theorem 9. $[N_\theta,F,p,\|,.,\|]^I$ *and* $[N_\theta,F,p,\|,.,\|]_0^I$ *are linear spaces.*

Proof. If

$$\left\{r\in\mathbb{N}:h_r^{-1}\sum_{k\in I_r}[f_k(\|x_k-L_1,z\|)]^{p_k}\geq\varepsilon\right\}\in I$$

and

$$\left\{r\in\mathbb{N}:h_r^{-1}\sum_{k\in I_r}[f_k(\|y_k-L_2,z\|)]^{p_k}\geq\varepsilon\right\}\in I.$$

Then we have

$$\begin{aligned} & h_r^{-1}\sum_{k\in I_r}[f_k(\|(\alpha_1x_k+\alpha_2y_k)-(\alpha_1L_1+\alpha_2L_2),z\|)]^{p_k} \\ \leq\ & Dh_r^{-1}T_{\alpha_1}^{\sup p_k}\sum_{k\in I_r}[f_k(\|x_k-L_1,z\|)]^{p_k} \\ & +Dh_r^{-1}T_{\alpha_2}^{\sup p_k}\sum_{k\in I_r}[f_k(\|y_k-L_2,z\|)]^{p_k} \end{aligned}$$

and

$$\begin{aligned} & \left\{r\in\mathbb{N}:h_r^{-1}\sum_{k\in I_r}[f_k(\|(\alpha_1x_k+\alpha_2y_k)-(\alpha_1L_1+\alpha_2L_2),z\|)]^{p_k}\right\} \\ \subset\ & \left\{r\in\mathbb{N}:Dh_r^{-1}T_{\alpha_1}^{\sup p_k}\sum_{k\in I_r}[f_k(\|x_k-L_1,z\|)]^{p_k}\right\} \\ & \cup\left\{r\in\mathbb{N}:Dh_r^{-1}T_{\alpha_2}^{\sup p_k}\sum_{k\in I_r}[f_k(\|y_k-L_2,z\|)]^{p_k}\right\}. \end{aligned}$$

This implies $\left\{r\in\mathbb{N}:h_r^{-1}\sum_{k\in I_r}[f_k(\|(\alpha_1x_k+\alpha_2y_k)-(\alpha_1L_1+\alpha_2L_2),z\|)]^{p_k}\geq\varepsilon\right\}\in I$. ∎

Theorem 10. *Let $F = (f_k)$ be a sequence of modulus functions, $(X, \|.,.\|)$ is a $2-$Banach space and (x_k) is lacunary strongly convergent to L in $[N_\theta, p, \|,.,\|]^I$ then (x_k) is lacunary strongly convergent to L in $[N_\theta, F, p, \|,.,\|]^I$.*

Proof. Now let $\varepsilon > 0$ be given. We can choose $0 < \delta < 1$ such that for every t with $0 \le t \le \delta$ we have $f(t) < \varepsilon$. Now using the previous lemma we get

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} [f_k (\|x_k - L, z\|)]^{p_k} \ge \varepsilon \right\}$$

$$= \left\{ r \in \mathbb{N} : h_r^{-1} \left(h_r \max \left\{ \varepsilon^{\inf p_k}, \varepsilon^{\sup_k p_k} \right\} \right) \ge \varepsilon \right\}$$

$$\cup \left\{ r \in \mathbb{N} h_r^{-1} \max \left\{ \left(2 \sup_k f_k (1) \delta^{-1} \right)^{\inf p_k}, \left(2 \sup_k f_k (1) \delta^{-1} \right)^{\sup_k p_k} \right\} \sum_{k \in I_r} [(\|x_k - L, z\|)]^{p_k} \right\}.$$

and this completes the proof. ∎

Theorem 11. *Let $(X, \|.,.\|)$ be a $2-$Banach space, $F = (f_k)$ be a sequence of modulus functions and $\lim_{t \to \infty} \inf_k \frac{f_k(u)}{u} > 0$ then $[N_\theta, F, p, \|,.,\|]^I = [N_\theta, p, \|,.,\|]^I$.*

Proof. The following inequality gives us the required.

$$\left\{ r \in \mathbb{N} : h_r^{-1} \sum_{k \in I_r} [f_k (\|x_k - L, z\|)]^{p_k} \ge \varepsilon \right\} \supseteq \left\{ r \in \mathbb{N} : a h_r^{-1} \sum_{k \in I_r} \|x_k - L, z\|^{p_k} \ge \varepsilon \right\},$$

where a is a positive number such that $\frac{f_k(u)}{u} > au$ for $u > 0$ and each $k \in \mathbb{N}$. ∎

References

[1] S. Pehlivan, B. Fisher, Lacunary strong convergence with respect to a sequence of modulus functions, *Comment. Math. Univ. Carolinae* 36,1(1995), 71-78.

[2] R. Colak, Lacunary strong convergence of difference sequences with respect to a modulus function, *Filomat, 17 (2003), 9-14*.

[3] B.C. TTripathy, S. Mahanta and M. Et, On generalized lacunary difference vector valued paranormed sequences defined by Orlicz functions, International journal of Mathemetical sciences, december 2005, Volume 4, No. 2, pp. 341-355.

[4] H. Fast, Sur la convergence statistique, *Colloq. Math.,* **2** (1951), 241 – 244.

[5] S. Gähler, 2−metrische Räume und ihre topologische Struktur, *Math. Nachr.*, **26** (1963), 115 – 148.

[6] H. Gunawan and Mashadi, On Finite Dimensional 2−normed spaces, *Soochow J. Math.*, **27**(3) (2001), 321 – 329.

[7] B.C. TTripathy, M. Et and Y. Altin, Generalized Difference Sequence Spaces Defined by Orlicz Function in a Locally Convex Space, *Journal of Analysis and Applications*, Vol. 1 (2003), No.3, pp. 175-192.

[8] A. Sahiner, M. Gurdal, S. Saltan and H. Gunawan, Ideal Convergence in 2-normed spaces, T*aiwanese Journal of Mathematics*, (to appear).

[9] I.J. Maddox, Elements of Functional Analysis, Cambridge Univ. Press, 1970.

[10] P. Kostyrko, M. Macaj and T. Salat, $I-$Convergence*, Real Anal. Exchange*, **26** (2) (2000) , 669 – 686.

[11] P. Kostyrko, M. Macaj, T. Salat and M. Sleziak, $I-$Convergence and Extremal $I-$Limit Points *(* to submit in *Math. Slovaca*).

[12] W. Raymond, Y. Freese and J. Cho, Geometry of linear 2−normed spaces, N.Y. Nova Science Publishers, Huntington, 2001.

[13] H. Steinhaus, Sur la convergence ordinarie et la convergence asymptotique, *Colloq. Math.,* **2** (1951), 73 − 74.

[14] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesaro-type summability spaces, *Proc. London Math. Soc.* 37 (3) (1978), 508-520.

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