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Generalized Cocompact Lattices

M. Motamedi

Department of Mathematics
Chamran University, Ahvaz, Iran
motamedi_m@scu.ac.ir

Abstract

We introduce the concept of λ-cocompact lattices and get some basic results which generalize those of cocompact lattices.

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1.Introduction

In [1] a lattice is called cocompact if each discover of 0 has a finite subdiscover, i.e for every subset X of L such that $\bigwedge X = 0$ there exists a finite subset F of X such that $\bigwedge F = 0$. The concept of a-Artinian modules is introduced in [4] extending some well-known results for Artinian modules. Our aim in this note is to prove most results of [4] in a lattical general setting, which at the same time extends some results of [1].

Throughout, L always denotes a modular upper-continuous lattice with the top element 1 and the bottom element $0 (0 \neq 1)$. For all $a \leq b$, $[a, b] = \{x \in L,\ a \leq x \leq b\}$ is called the factor of b by a. Obviously $[a, b]$ is a sublattice of L and $\phi_b : x \longrightarrow x \vee b$ is an isomorphism of $[a, a \vee b]$ onto $[a \wedge b, b]$, the inverse isomorphism is $\Psi_a : x \longrightarrow x \wedge a$. For all the notions, such as essential, pseudo-complement in a lattice and notation we refer to [2] and [3]. we use $|I|$, ω to denote the cardinality of a set I, and the ordinal of natural numbers.

2.Results

First we recall the elefinition of a regular cardinal.

Definition 2.1. An infinite cardinal λ is called regular if for every indexed set I with $|I| < \lambda$ and for any nonempty set $\{A_i : i \in I\}$ with $|A_i| < \lambda$, we have $|\bigcup_{i\in I} A_i| < \lambda$.

The following definition extends the concept of cocompactnes.

Definition 2.2. Let λ be a regular cardinal. A lattice L is called λ-cocompact if 0 has a subdiscover of cardinality less than λ, i.e. for any set $\{a_i : i \in I\}$ of elements of L with $\bigwedge_{i\in I} a_i = 0$, there exists a subset I_0 of I with $|I_0| < \lambda$ such that, $\bigwedge_{i\in I'} a_i = 0$, and λ is the least regular cardinal with this property.

Example 2.3. Cocompact lattices are indeed ω-cocompact lattices.

Lemma 2.4. Let L be a lattice and $a \in L$, then $[a, 1]$ is λ-cocompact if for any set $\{a_i : i \in I\}$ of elements of $[a, 1]$ with $\bigwedge_{i\in I} a_i = a$, there exists $I_0 \subseteq I$ with $|I_0| < \lambda$ such that $\bigwedge_{i\in I_0} a_i = a$ and λ is the least regular cardinal with this property.

Proof: Evident.

Corollary 2.5. Let L be a lattice, if for every $0 \neq a \in L$, $[a, 1]$ is $\lambda-$cocompact, then for any set $\{a_i : i \in I\}$ of elements of L there exists $I_0 \subset I$ with $|I_0| < \lambda$ such that $\bigwedge_{i\in I} a_i = \bigwedge_{i\in I_0} a_i$.

Proof: Put $a = \bigwedge_{i\in I} a_i$ in the above lemma.

The following is an extension of proposition 2.2 in [1].

Proposition 2.6. Let L be a lattice and $0 \neq a \in L$, then $[0, a]$ and $[a, 1]$ are λ_1 and λ_2-cocompact for some $\lambda_1 \leq \lambda$ and $\lambda_2 \leq \lambda$. If for some $0 \neq a \in L$, $[0, a]$ is λ_1-cocompact and $[a, 1]$ is λ_2-cocompact, then L is λ-cocompact, where $\lambda = Max(\lambda_1, \lambda_2)$.

Proof: The first statment is evident. Let $\bigwedge_{i\in I} b_i = 0$ be a discover of 0 in L. Then $\bigwedge(a \wedge b_i) = a \wedge (\bigwedge_{i\in I} b_i) = a \wedge 0 = 0$ is a discover of 0 in $[0, a]$. As $[0, a]$ is λ_1-cocompact there exists a set $I_1 \subseteq I$ with $|I_1| < \lambda$ such that

$0 = \bigwedge_{i \in I_1}(a \wedge b_i) = a \wedge (\bigwedge_{i \in I_1} b_i)$. If $\bigwedge_{i \in I_1} b_i = 0$, the proof is complete. If $\bigwedge_{i \in I_1} b_i \neq 0$. then let c be a pseudo-complement of a which contains $\bigwedge_{i \in I_1} b_i$. We have $\bigwedge_{i \in I_1} b_i \in [0, c] = [a \wedge c, c] \cong [a, a \vee c] \subseteq [a, 1]$. The lattice $[a, 1]$ being λ_2-cocompact, $[a, a \vee c]$ and hence $[0, c]$ is μ-cocompact for some $\mu \leq \lambda_2$. Now $0 = \bigwedge_{i \in I}(c \wedge b_i)$ being a discover of 0 in $[0, c]$, there is a set $I_2 \subseteq I$ with $|I_2| < \mu$ such that $0 = \bigwedge_{i \in I_2}(c \wedge b_i) = c \wedge (\bigwedge_{i \in I_2} b_i)$. Now for $b = \bigwedge_{i \in I_1 \cup I_2} b_i$ we have $b \leq \bigwedge_{i \in I_1} b_i \leq c$ and $c \wedge b \leq c \wedge (\bigwedge_{i \in I_2} b_i) = 0$ so that $b = 0$, and we have the required subdiscover, as $|I_1 \cup I_2| = Max(|I_1|, |I_2|) \leq Max(\lambda_1, \lambda_2)$.

Corollary 2.7. If L is a lattice and $0 \neq a$ is such that both $[0, a]$ and $[a, 1]$ are cocompact then L is cocompact.

Proof. Put $\lambda_1 = \lambda_2 = \omega$, in the proposition 2.6

Corollary 2.8. Let L be a λ-cocompact lattice. If $a \in L$ is essential in L, then $[0, a]$ is also λ-cocompact.

Proof: $[0, a]$ is μ-cocompact for some $\mu \leq \lambda$. Suppose $\mu < \lambda$ and seek a contradiction. Let $\{a_i : i \in I\}$ be an arbitrary set of nonzero elements of L such that $\bigwedge_{i \in I} a_i = 0$. The element a being essential in L, we have $a \wedge a_i \neq 0$ for all i and $0 = a \wedge (\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I}(a \wedge a_i)$. Hence $\{a \wedge a_i : i \in I\}$ is a discover of 0 in $[0, a]$ and $[0, a]$ begin μ-cocompact, there exists $I' \subseteq I$ with $|I'| < \mu$ and $0 = \bigwedge_{i \in I'}(a \wedge a_i) = a \wedge (\bigwedge_{i \in I'} a_i)$. But again essentiality of a implies $\bigwedge_{i \in I'} a_i = 0$ in L, a contradiction as $|I'| < \mu < \lambda$ and by assumption λ is the least regular cardinal with this property.

We need the following definition to be able to define λ-Artinian lattices.

Definition 2.9. A partially ordered relation on a set A is called cowell-ordred if every nonempty subset of A has a greatest element.

Definition 2.10. Let λ be a regular cardinal. A lattice L is called λ-Artinian if for every cowell-ordred descending chain $\{a_\alpha : \alpha < \lambda\}$ of elements of L, there exists $\gamma < \lambda$ such that for all $\beta \geq \gamma$, $a_\gamma = a_\beta$ and λ is the least regular cardinal with this property.

Remark: For every limit ordinal $\gamma < \lambda$, we put $a_\gamma = \bigwedge_{\beta < \lambda} a_\beta$.

Remark: As all the chains of a lattice form a set, every lattice is λ-Artinian for some λ.

Example 2.11. A lattice L is Artinian if and only it is ω-Artinian.

It is shown in [1] that a lattice is Artinain if and only if for every $a \neq 1$, $[a,1]$ is cocompact Next we extend this result to an arbitrary λ-Artinian lattice.

Theorem 2.12. A lattice L is λ-Artinain if and only if for any $1 \neq a \in L$, $[a,1]$ is λ-cocompact.

Proof: Let L be a λ-Artinain lattice, we have to show that for all $a \neq 1$, $[a,1]$ is λ-cocompact, so let $\{a_i : \ i \in I\}$ be a set of elements of R with $\bigwedge_{i\in I} a_i = a$ and suppose the index set is well-ordered. Now the set $\{\bigwedge_{i\geq\alpha} a_i : \ i \in I, \alpha < \lambda\}$ is a cowell-ordred descending chain of elements in L. L being Artinian , there exists $\gamma < \lambda$ such that for all $\beta \geq \gamma$, $\bigwedge_{i\geq\gamma} a_i = \bigwedge_{i\geq\beta} a_i$. It follows that there exists a subset I' of I with $|I'| < \lambda$ such that $\bigwedge_{i\in I'} a_i = \bigwedge_{i\in I} a_i = a$. As the proof of the converse shows λ is the least regular cardinal with this property.

Conversly let $[0.1]$ be λ-cocompact for all $1 \neq a \in L$ and let $\{a_\alpha : \ \alpha < \lambda\}$ be an arbitrary cowell-ordred descending chain in L. By assumption $[\bigwedge_{\alpha<\lambda} a_i, 1]$ is λ-cocompact, so there exists a subset B of A such that $|B| < \lambda$ and $\bigwedge_{\alpha\in B} a_\alpha = \bigwedge_{\alpha\in A} a_\alpha$. Now it follows that L is λ_1-Artinian for some $\lambda_1 \leq \lambda$. But the proof of the first part can be applied to show that $\lambda_1 < \lambda$ is not possible.

Proposition 2.13. Let L be a lattice, with $1 \neq a \in L$. If $[0,a]$ is λ_1-Artinian and $[0,1]$ is λ_2-Artinian , then L is λ-Artinian where $\lambda = Max(\lambda_1, \lambda_2)$.

Proof: We have to show for every $0 < c < 1$, $[c,1]$ is λ-cocompact. consider $a \vee c$ and $a \wedge c$. By the theorem $[a \wedge c, a]$ is λ_1-cocompact and $[a \vee c, 1]$ is λ_2-cocompact. Now $[a \wedge c, a] \cong [c, a \vee c]$ implies $[c, a \vee c]$ is λ_1-cocompact, and $[a \vee c, 1]$ is λ_2-cocompact. therefore $[c,1]$ is $Max(\lambda_1, \lambda_2) = \lambda$-cocompact. Now we are through.

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