Int. J. Contemp. Math. Sci., Vol. 2, 2007, no. 17, 831 - 837

A Note on the Sequence $\sum_{i=1}^{n} \frac{1}{c_{i,k}}$ where $c_{n,k}$ Have Just k Prime Factors

Rafael Jakimczuk

Division Matematica, Universidad Nacional de Lujan
Buenos Aires, Argentina
jakimczu@mail.unlu.edu.ar

Abstract

In this article we prove the formulas

$$\sum_{i=1}^{n} \frac{1}{c_{i,k}} = \frac{(\log\log n)^k}{k!} + O((\log\log n)^{k-1})$$

$$\sum_{c_{i,k} \leq x} \frac{1}{c_{i,k}} = \frac{(\log\log x)^k}{k!} + O((\log\log x)^{k-1})$$

where $c_{n,k}$ is the sequence of numbers which have just k prime factors in their factorization ($k \geq 2$).

Mathematics Subject Classification: 11N05, 11N37

Keywords: Numbers which have just k prime factors in their factorization, divergent series, asymptotic formulas.

1 Introduction and lemmas

Let $c_{n,k}$ be the sequence of numbers which have just k prime factors in their factorization ($k \geq 1$). If $k = 1$ then $c_{n,1} = p_n$ is the sequence of prime numbers.

In a previous article [2] we have proved

$$\sum_{i=1}^{n} \frac{1}{c_{i,k}} \sim \frac{(\log\log n)^k}{k!}$$

$$\sum_{c_{i,k} \leq x} \frac{1}{c_{i,k}} \sim \frac{(\log\log x)^k}{k!}$$

In the same way we can prove that

$$\sum_{i=1}^{n} \frac{\log c_{i,k}}{c_{i,k}} \sim \frac{\log n (\log\log n)^{k-1}}{(k-1)!}$$

$$\sum_{c_{i,k} \leq x} \frac{\log c_{i,k}}{c_{i,k}} \sim \frac{\log x (\log\log x)^{k-1}}{(k-1)!}$$

If $k = 1$ these formulas are well known [1].

In this article we prove more precise formula

$$\sum_{c_{i,k} \leq x} \frac{1}{c_{i,k}} = \frac{(\log\log x)^k}{k!} + O((\log\log x)^{k-1}) \qquad (k \geq 2)$$

In addition we also prove

$$\sum_{c_{i,k} \leq x} \frac{\log c_{i,k}}{c_{i,k}} = \frac{\log x (\log\log x)^{k-1}}{(k-1)!} + O(\log x (\log\log x)^{k-2}) \qquad (k \geq 2)$$

The following theorems are well known [1], we shall use them as lemmas.

Lemma 1.1 *The following formula holds*

$$\sum_{p_i \leq x} \frac{\log p_i}{p_i} = \log x + f(x)$$

Where $f(x) = O(1)$, therefore $|f(x)| < K$ if $x \geq 1$.

Lemma 1.2 *The following formula holds*

$$\sum_{p_i \leq x} \frac{1}{p_i} = \log\log x + O(1)$$

Lemma 1.3 *Suppose that $a_1, a_2, a_3, \ldots$ is a sequence of numbers, that*

$$C(t) = \sum_{n \leq t} a_n \tag{1}$$

If $a_j = 0$ for $j < n_1$ and $f(t)$ has a continuous derivative for $t \geq n_1$, then

$$\sum_{n \leq x} a_n f(n) = C(x) f(x) - \int_{n_1}^{x} C(t) f'(t)\, dt \tag{2}$$

2 Main Results

Theorem 2.1 *If $k \geq 2$ the following formulas hold*

$$\sum_{c_{i,k}\leq x} \frac{1}{c_{i,k}} = \frac{(\log\log x)^k}{k!} + O((\log\log x)^{k-1}) \tag{3}$$

$$\sum_{c_{i,k}\leq x} \frac{\log c_{i,k}}{c_{i,k}} = \frac{\log x(\log\log x)^{k-1}}{(k-1)!} + O(\log x(\log\log x)^{k-2}) \tag{4}$$

Proof. We shall apply mathematical induction. If $k = 2$ we obtain (lemma 1.1 and lemma 1.2)

$$\sum_{c_{i,2}\leq x} \frac{\log c_{i,2}}{c_{i,2}} = \sum_{p_i\leq x}\left(\frac{1}{p_i}\sum_{p_j\leq \frac{x}{p_i}} \frac{\log p_j}{p_j}\right) + \sum_{p_i^2\leq x}\frac{\log p_i}{p_i^2} =$$

$$\sum_{p_i\leq x}\frac{1}{p_i}\left(\log\left(\frac{x}{p_i}\right) + f\left(\frac{x}{p_i}\right)\right) + O(1) = \log x\sum_{p_i\leq x}\frac{1}{p_i} -$$

$$\sum_{p_i\leq x}\frac{\log p_i}{p_i} + \sum_{p_i\leq x}\frac{1}{p_i} f\left(\frac{x}{p_i}\right) + O(1) =$$

$$\log x(\log\log x + O(1))$$

$$-\log x + \sum_{p_i\leq x}\frac{1}{p_i} f\left(\frac{x}{p_i}\right) + O(1) = \log x\log\log x + O(\log x)$$

Since (lemma 1.1)

$$\left|\sum_{p_i\leq x}\frac{1}{p_i} f\left(\frac{x}{p_i}\right)\right| \leq \sum_{p_i\leq x}\frac{1}{p_i}\left|f\left(\frac{x}{p_i}\right)\right| \leq K\sum_{p_i\leq x}\frac{1}{p_i} = K(\log\log x + O(1))$$

Therefore

$$\sum_{p_i\leq x}\frac{1}{p_i} f\left(\frac{x}{p_i}\right) = O(\log\log x)$$

In lemma 1.3 let us put $a_{c_{i,2}} = \frac{\log c_{i,2}}{c_{i,2}}$ and $a_n = 0$ if n has not two prime factors. Then (1) becomes

$$C(t) = \sum_{c_{i,2}\leq t}\frac{\log c_{i,2}}{c_{i,2}} = \log t\log\log t + O(\log t)$$

With $f(t)=1/\log t$, (2) becomes

$$\begin{aligned}\sum_{c_{i,2}\le x}\frac{1}{c_{i,2}}=\frac{C(x)}{\log x}+\int_4^x\frac{C(t)}{t\log^2 t}\,dt &= \log\log x+O(1)+\int_4^x\frac{\log\log t}{t\log t}\,dt\\ +\int_4^x\frac{O(\log t)}{t\log^2 t}\,dt &= \log\log x+O(1)\\ +\frac{(\log\log x)^2}{2}+O(\log\log x) &= \frac{(\log\log x)^2}{2}+O(\log\log x)\end{aligned}$$

Therefore, the theorem is true if $k=2$. Suppose the theorem is true for $k-1$, that is (inductive hypothesis)

$$\sum_{c_{i,k-1}\le x}\frac{\log c_{i,k-1}}{c_{i,k-1}}=\frac{\log x(\log\log x)^{k-2}}{(k-2)!}+O(\log x(\log\log x)^{k-3}) \qquad (k\ge 3) \quad (5)$$

$$\sum_{c_{i,k-1}\le x}\frac{1}{c_{i,k-1}}=\frac{(\log\log x)^{k-1}}{(k-1)!}+O((\log\log x)^{k-2}) \qquad (k\ge 3) \qquad (6)$$

We shall prove that the theorem is also true for k. Let $c'_{n,k-1}$ the subsequence of the numbers $c_{n,k-1}$ which have at least two prime factors equals. Then we have

$$\begin{aligned}0\le\sum_{c'_{i,k-1}\le x}\left(\frac{1}{c'_{i,k-1}}\sum_{p_j\le\frac{x}{c'_{i,k-1}}}\frac{k\log p_j}{p_j}\right)\le k\left(\sum_{c'_{i,k-1}\le x}\frac{1}{c'_{i,k-1}}\right)\left(\sum_{p_j\le x}\frac{\log p_j}{p_j}\right) &\le\\ k\left(\sum_{p_j^2\le x}\left(\frac{1}{p_j^2}\sum_{c_{i,k-3}\le\frac{x}{p_j^2}}\frac{1}{c_{i,k-3}}\right)\right)\left(\sum_{p_j\le x}\frac{\log p_j}{p_j}\right) &\le\\ k\left(\sum_{p_j\le x}\frac{1}{p_j^2}\right)\left(\sum_{c_{i,k-3}\le x}\frac{1}{c_{i,k-3}}\right)\left(\sum_{p_j\le x}\frac{\log p_j}{p_j}\right) &\le\\ kC\left(\frac{(\log\log x)^{k-3}}{(k-3)!}+O((\log\log x)^{k-4})\right)(\log x+O(1))\end{aligned}$$

with $C>0$. That is

$$\sum_{c'_{i,k-1}\le x}\left(\frac{1}{c'_{i,k-1}}\sum_{p_j\le\frac{x}{c'_{i,k-1}}}\frac{k\log p_j}{p_j}\right)=O(\log x(\log\log x)^{k-3}) \qquad (7)$$

Let us consider the numbers $c_{n,k}$ of the form $p_1^2p_2\ldots p_{k-1}$ where $p_1,p_2\ldots p_{k-1}$ are distinct primes. Then we have

$$0 \leq \sum_{p_1^2p_2\ldots p_{k-1}\leq x}\frac{\log p_1}{p_1^2p_2\ldots p_{k-1}} \leq \sum_{p_1^2\leq x}\left(\frac{\log p_1}{p_1^2}\sum_{p_2\ldots p_{k-1}\leq \frac{x}{p_1^2}}\frac{1}{p_2\ldots p_{k-1}}\right)$$
$$\leq \sum_{p_1^2\leq x}\left(\frac{\log p_1}{p_1^2}\sum_{c_{k-2,i}\leq x}\frac{1}{c_{k-2,i}}\right) \leq \left(\sum_{p_1\leq x}\frac{\log p_1}{p_1^2}\right)\left(\sum_{c_{k-2,i}\leq x}\frac{1}{c_{k-2,i}}\right) \leq$$
$$D\left(\frac{(\log\log x)^{k-2}}{(k-3)!}+O((\log\log x)^{k-3})\right)$$

with $D>0$. That is

$$\sum_{p_1^2p_2\ldots p_{k-1}\leq x}\frac{\log p_1}{p_1^2p_2\ldots p_{k-1}} = O((\log\log x)^{k-2}) \tag{8}$$

Now, we have

$$\sum_{c_{i,k}\leq x}\frac{\log c_{i,k}}{c_{i,k}} = \sum_{c_{i,k-1}\leq x}\left(\frac{1}{c_{i,k-1}}\sum_{p_j\leq \frac{x}{c_{i,k-1}}}\frac{\log p_j}{p_j}\right)+F(x) \tag{9}$$

where

$$F(x) \leq \sum_{c'_{i,k-1}\leq x}\left(\frac{1}{c'_{i,k-1}}\sum_{p_j\leq \frac{x}{c'_{i,k-1}}}\frac{k\log p_j}{p_j}\right)+\sum_{p_1^2p_2\ldots p_{k-1}\leq x}\frac{\log p_1}{p_1^2p_2\ldots p_{k-1}} \tag{10}$$

Equations (10), (7) and (8) give

$$F(x) = O(\log x(\log\log x)^{k-3}) \tag{11}$$

Therefore (9) is (lemma 1.2, (5), (6) and (11))

$$\sum_{c_{i,k}\leq x}\frac{\log c_{i,k}}{c_{i,k}} = \sum_{c_{i,k-1}\leq x}\left(\frac{1}{c_{i,k-1}}\sum_{p_j\leq \frac{x}{c_{i,k-1}}}\frac{\log p_j}{p_j}\right)+F(x) =$$
$$\sum_{c_{i,k-1}\leq x}\left(\frac{1}{c_{i,k-1}}\left(\log\left(\frac{x}{c_{i,k-1}}\right)+f\left(\frac{x}{c_{i,k-1}}\right)\right)\right)+F(x) =$$
$$\log x\sum_{c_{i,k-1}\leq x}\frac{1}{c_{i,k-1}}-\sum_{c_{i,k-1}\leq x}\frac{\log c_{i,k-1}}{c_{i,k-1}}+\sum_{c_{i,k-1}\leq x}\left(\frac{1}{c_{i,k-1}}f\left(\frac{x}{c_{i,k-1}}\right)\right)+F(x) =$$
$$\log x\left(\frac{(\log\log x)^{k-1}}{(k-1)!}+O((\log\log x)^{k-2})\right)-\frac{\log x(\log\log x)^{k-2}}{(k-2)!} +$$
$$O(\log x(\log\log x)^{k-3})+O((\log\log x)^{k-1})+F(x) =$$
$$\frac{\log x(\log\log x)^{k-1}}{(k-1)!}+O(\log x(\log\log x)^{k-2})$$

That is (4).

In lemma 1.3 let us put $a_{c_{i,k}} = \frac{\log c_{i,k}}{c_{i,k}}$ and $a_n = 0$ if n has not k prime factors. Then (1) becomes

$$C(t) = \sum_{c_{i,k} \leq t} \frac{\log c_{i,k}}{c_{i,k}} = \frac{\log t (\log \log t)^{k-1}}{(k-1)!} + O(\log t (\log \log t)^{k-2})$$

With $f(t) = 1/\log t$, (2) becomes

$$\begin{aligned} \sum_{c_{i,k} \leq x} \frac{1}{c_{i,k}} &= \frac{C(x)}{\log x} + \int_{2^k}^{x} \frac{C(t)}{t \log^2 t}\, dt = \frac{(\log \log x)^{k-1}}{(k-1)!} + O((\log \log x)^{k-2}) \\ &+ \frac{1}{(k-1)!} \int_{2^k}^{x} \frac{(\log \log t)^{k-1}}{t \log t}\, dt + \int_{2^k}^{x} \frac{O(\log t (\log \log t)^{k-2})}{t \log^2 t}\, dt = \frac{(\log \log x)^{k-1}}{(k-1)!} \\ &+ O((\log \log x)^{k-2}) + \frac{(\log \log x)^k}{k!} + O((\log \log x)^{k-1}) = \frac{(\log \log x)^k}{k!} + O((\log \log x)^{k-1}) \end{aligned}$$

That is (3). Thus the theorem is proved.

Note in this proof we do not need any assumption on the distribution of the numbers $c_{n,k}$. Our argument is strictly combinatorial. Let $\pi_k(x)$ be the number of these numbers not exceeding x, it was proved by Landau [1] that

$$\pi_k(x) \sim \frac{x (\log \log x)^{k-1}}{(k-1)! \log x}$$

On the other hand, in the following theorem, we need $\pi_k(x)$.

Theorem 2.2 *The following formulas hold*

$$\sum_{i=1}^{n} \frac{1}{c_{i,k}} = \frac{(\log \log n)^k}{k!} + O((\log \log n)^{k-1}) \tag{12}$$

$$\sum_{i=1}^{n} \frac{\log c_{i,k}}{c_{i,k}} = \frac{\log n (\log \log n)^{k-1}}{(k-1)!} + O(\log n (\log \log n)^{k-2}) \tag{13}$$

Proof. In [2] we proved that

$$c_{n,k} \sim \frac{(k-1)! n \log n}{(\log \log n)^{k-1}} \tag{14}$$

Equation (14) gives

$$\log c_{n,k} = \log(k-1)! + \log n + \log\log n - (k-1)\log\log\log n + o(1) \tag{15}$$

Equation (15) gives

$$\log c_{n,k} \sim \log n \tag{16}$$

$$\frac{\log c_{n,k}}{\log n} = 1 + o\left(\frac{1}{\log\log n}\right) \tag{17}$$

Equation (16) gives

$$\log\log c_{n,k} = \log\log n + o(1) \tag{18}$$

that is

$$\frac{(\log\log c_{n,k})^k}{(\log\log n)^k} = 1 + o\left(\frac{1}{\log\log n}\right) \tag{19}$$

Equations (3), (18) and (19) give

$$\begin{aligned}\sum_{i=1}^{n}\frac{1}{c_{i,k}} = \sum_{c_{i,k}\le c_{n,k}}\frac{1}{c_{i,k}} &= \frac{(\log\log c_{n,k})^k}{k!} + O((\log\log c_{n,k})^{k-1}) \\ &= \frac{(\log\log n)^k}{k!} + O((\log\log n)^{k-1})\end{aligned}$$

That is (12) . Equations (17) and (19) give

$$\frac{\log c_{n,k}}{\log n}\frac{(\log\log c_{n,k})^{k-1}}{(\log\log n)^{k-1}} = 1 + o\left(\frac{1}{\log\log n}\right) \tag{20}$$

Equations (4), (16), (18) and (20) give

$$\begin{aligned}\sum_{i=1}^{n}\frac{\log c_{i,k}}{c_{i,k}} = \sum_{c_{i,k}\le c_{n,k}}\frac{\log c_{i,k}}{c_{i,k}} &= \frac{\log c_{n,k}(\log\log c_{n,k})^{k-1}}{(k-1)!} + \\ O(\log c_{n,k}(\log\log c_{n,k})^{k-2}) &= \frac{\log n(\log\log n)^{k-1}}{(k-1)!} + \\ +O(\log n(\log\log n)^{k-2})\end{aligned}$$

That is (13). The theorem is proved.

References

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Received: December 11, 2006