

An Explicit Formula and Estimations for Hecke L -Functions: Applying the Li Criterion

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Abstract

We advance the use of a criterion for the validity of the generalized Riemann hypothesis. We rewrite and then estimate various terms of an explicit formula for Hecke L -functions very recently obtained by Li. We demonstrate how additional information on a coefficient function appearing in the logarithmic derivative of the L function would lead directly to verification of a generalized Riemann hypothesis. More generally, we mention a direct link between the generalized Ramanujan conjectures and the validity of generalized Riemann hypotheses.

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Introduction

The generalized Riemann hypothesis [29] is well recognized to be one of the outstanding problems of mathematical physics [22]. The appearance of the associated generalized zeta functions occur in diverse areas of physics including quantum gravity and quantum field theory [15] and thermodynamics and quantum dynamical systems [25, 34, 35, 3, 4]. For instance, the complex zeros of the zeta function provide a model to test general theories of quantum chaotic systems [34, 35].

In this paper we are concerned with an equivalence [28] of the generalized Riemann hypothesis for Hecke L -functions [16, 7, 22, 30, 18]. In particular, all complex zeros of a generalized ξ function, $\xi_N(s)$, lie on the critical line $\text{Re } s = 1/2$ if and only if $\tau_N(n) \geq 0$ for all integers $n = 1, 2, \dots$ [28]. Here,

$\tau_N(n)$ are certain real constants to be defined shortly. They generalize the Li/Keiper constants [23, 26, 27, 6, 8, 9] corresponding to the Riemann xi function [29, 17, 33, 13, 21, 11]. In 1992 Keiper introduced his form of these constants and provided numerical results.

We adopt the notation of Ref. [28] in order to minimize background information and other explanatory material. We first re-express and then estimate the major result of this reference (Theorem 1.4). Much of this process extends our previous work [9] for the classical zeta function and for classical L functions. We are then able to show a close link between the generalized Ramanujan conjectures [30, 20] and the generalized Riemann hypothesis.

We let [28]

$$\xi_N(s) = N^{gs/2} (2\pi)^{-gs} \Gamma^g \left(s + \frac{k-1}{2} \right) L_N \left(s + \frac{k-1}{2} \right), \quad (1)$$

where the positive integer weight $k > 2$, g is the dimension of the space $S_k(N, \chi)$, Γ is the Gamma function, the Hecke L -function L_N has an Euler product that we omit, and χ is a Dirichlet character of modulus N with $\chi(-1) = (-1)^k$. Here, $S_k(N, \chi)$ is the space of all cusp forms of weight k and character χ for the Hecke congruence subgroup $\Gamma_0(N)$ of level N . In particular, $f \in S_k(N, \chi)$ is holomorphic in the upper half plane and transforms as $f[(az+b)/(cz+d)] = \chi(d)(cz+d)^k f(z)$ for all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

where [19] $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z) \mid c \equiv 0 \pmod{N} \right\}$.

Then [28]

$$\tau_N(n) = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho} \right)^{-n} \right], \quad n = 1, 2, \dots, \quad (2)$$

where the sum is over all zeros ρ of $\xi_N(s)$ taken in the order of increasing imaginary part. The symmetry $\tau_N(n) = \tau_N(-n)$ holds due to the functional equation $\xi_N(s) = w \bar{\xi}_N(1 - \bar{s})$, where w is a constant, and the fact that the sum in Eq. (2) is over all zeros.

We put

$$\mathfrak{f}_N \equiv N^{\nu_N} \prod_{\mathfrak{f} \mid m, m \mid N} m^{\nu_m d(N/m)}, \quad (3)$$

where \mathfrak{f} is the conductor and d is the divisors function. For $\mathfrak{f} \mid m$ and $m \mid N$, ν_m denotes the dimension of the subspace generated by all newforms in $S_k(m, \chi_m)$, where χ_m is the Dirichlet character of modulus m induced by the Dirichlet character χ . In terms of traces of the Hecke operator T , we put $B(p^\ell) = \text{tr}[T(p^\ell)] - \chi(p)p^{k-1} \text{tr}[T(p^{\ell-2})]$, for p not dividing N and p a prime number.

Then Theorem 1.4 of Ref. [28] can be written as

$$\begin{aligned} \tau_N(n) &= \frac{n}{2} \ln f_N - ng \left(\ln 2\pi + \gamma + \frac{2}{k+1} \right) \\ &- \sum_{\ell=1}^n \binom{n}{\ell} \frac{(-1)^{\ell-1}}{(\ell-1)!} \sum_{m=1, (m,N)=1}^{\infty} \frac{\Lambda(n)}{m^{(k+1)/2}} B(m) \ln^{\ell-1} m \\ &+ ng \sum_{\ell=1}^{\infty} \frac{k+1}{\ell(2\ell+k+1)} + g \sum_{m=2}^n \binom{n}{m} \sum_{\ell=1}^{\infty} \frac{(-1)^m}{\left(\ell + \frac{k-1}{2}\right)^m}, \end{aligned} \tag{4}$$

where Λ is the von Mangoldt function and $\gamma \simeq 0.577216$ is the Euler constant.

Alternative representation and estimation of $\tau_N(n)$

We first show

Theorem 1

$$\begin{aligned} \tau_N(n) &= gn \left[\frac{1}{2g} \ln f_N + \psi \left(\frac{k+1}{2} \right) - \ln 2\pi \right] - \sum_{m=2, (m,N)=1}^{\infty} \frac{\Lambda(m)}{m^{(k+1)/2}} B(m) L_{n-1}^1(\ln m) \\ &+ g \sum_{\ell=1}^{\infty} \left[\left(\frac{\ell + \frac{k-3}{2}}{\ell + \frac{k-1}{2}} \right)^n + \frac{n}{\ell + \frac{k-1}{2}} - 1 \right], \end{aligned} \tag{5}$$

where L_{n-1}^1 is an associated Laguerre polynomial of degree $n-1$ and ψ is the digamma function. The case of weight $k=2$ was treated in Appendix E of Ref. [9].

Proof. The last sum in Eq. (5) follows from the last term in Eq. (4) upon an interchange of summations and use of the binomial expansion:

$$\sum_{m=2}^n \binom{n}{m} \left(\frac{-1}{\ell + \frac{k-1}{2}} \right)^m = \left(1 - \frac{1}{\ell + \frac{k-1}{2}} \right)^n + \frac{n}{\ell + \frac{k-1}{2}} - 1. \tag{6}$$

The sum with the von Mangoldt function in Eq. (5) follows from that in Eq. (4) upon interchange of summations, application of the defining power series for the associated Laguerre polynomial [14, 9] and use of the fact $\Lambda(1) = 0$. The rest of the terms in Eq. (5) follow from the identity [14]

$$\psi(x) + \gamma = x \sum_{k=1}^{\infty} \frac{1}{k(x+k)} - \frac{1}{x}, \tag{7}$$

and the functional equation for the digamma function, $\psi(x+1) = \psi(x) + 1/x$ for $x \neq 0$.

We next show

Theorem 2 The dominant behavior of the last term of Eq. (5) for $\tau_N(n)$ is $O(n \ln n)$.

For proof, we consider some comparison integrals for the last term of Eq. (5). We have

$$\begin{aligned} \int_1^\infty \left[\left(1 - \frac{1}{\ell + \frac{k-1}{2}} \right)^n + \frac{n}{\ell + \frac{k-1}{2}} - 1 \right] d\ell &= \left(\int_1^\infty - \int_1^{(k+1)/2} \right) \left[\frac{(x-1)^n}{x^n} + \frac{n}{x} - 1 \right] dx \\ &= n[\psi(n) + \gamma - 1] + 1 - \int_1^{(k+1)/2} \left[\frac{(x-1)^n}{x^n} + \frac{n}{x} - 1 \right] dx. \end{aligned} \quad (8)$$

The latter integral can be evaluated exactly with a change of variable,

$$\begin{aligned} \int_1^{(k+1)/2} \left[\frac{(x-1)^n}{x^n} + \frac{n}{x} - 1 \right] dx &= \left(\frac{k-1}{2} \right)^{n+1} \frac{1}{(n+1)} {}_2F_1 \left(n, n+1; n+2; \frac{1-k}{2} \right) \\ &\quad + \frac{1}{8}(k-1)(k+3)n - \frac{(k-1)}{2}, \end{aligned} \quad (9)$$

where ${}_2F_1$ is the hypergeometric function [1, 14, 5]. The right side of Eq. (9) vanishes when $k \rightarrow 1$, as it must. A separate examination of the ${}_2F_1$ factor (see the Appendix) shows that the first line of the right side of this equation contains a term $-n \ln[(k-1)/2]$. In fact, we have the estimation

$$\int_1^{(k+1)/2} \left[\frac{(x-1)^n}{x^n} + \frac{n}{x} - 1 \right] dx \leq n \ln \left(\frac{k+1}{2} \right) - \ln \left(\frac{k+1}{2} \right). \quad (10)$$

Therefore the integral in the right side of Eq. (8) contributes an $O(n)$ term. Combining Eq. (8) with (10) and using a standard inequality for the digamma function [9] gives the Theorem.

Remark. Additionally, we may comment on a portion of the integral in Eq. (9). If we put

$$I_n \equiv \int_1^{(k+1)/2} \left(\frac{x-1}{x} \right)^n dx = \int_0^{(k-1)/2} \frac{w^n}{(w+1)^n} dw, \quad n \geq 1, \quad (11)$$

we derive the property

$$I_n = \frac{n}{n-1} I_{n-1} + \frac{k-1}{2(-n+1)} \left(\frac{k-1}{k+1} \right)^{n-1}, \quad n > 0, \quad (12)$$

with $I_1 = (k-1)/2 - \ln[(k+1)/2]$. The homogeneous solution of the recursion relation (12) is obviously n . A particular solution can be developed, giving the general solution, which we too omit.

We next show that if we had a suitable bound on the function $B(m)$, the extended Riemann hypothesis would follow. This would hold because we would then have a lower bound on $\tau_N(n)$ becoming positive for sufficiently large n and computation would deliver the initial set of verified $\tau_N(j) \geq 0$, for $j = 1, \dots, j_0$.

We have a representative

Theorem 3 If $B(m) = O(m^{1/2-\varepsilon})$ (or smaller), where $\varepsilon > 0$ is arbitrary, then $\tau_N(n)$ is positive for all sufficiently large n .

We emphasize that we are not claiming that B necessarily has the size stated in the premise, but are illustrating that such a bound leads to information on the Li constants. For proof, we must show that the summation term in Eq. (5) with $\Lambda(m)B(m)$ is dominated by the leading behavior $O(n \ln n)$. In fact, we have the classical Szegő result [32] $|L_{n-1}^1(x)| \leq n \exp(x/2)$ so that $|L_{n-1}^1(\ln m)| \leq nm^{1/2}$. Then

$$\left| \sum_{m=2, (m,N)=1}^{\infty} \frac{\Lambda(m)}{m^{(k+1)/2}} B(m) L_{n-1}^1(\ln m) \right| \leq n \left| \sum_{m=2, (m,N)=1}^{\infty} \frac{\Lambda(m)}{m^{k/2}} B(m) \right|. \quad (13)$$

Since $k \geq 3$, the theorem follows.

Remarks. Under the stated premise, this Theorem shows that the summation term of interest is $O(n)$ or smaller. We can next heuristically argue that the optimal order of this term is $O(n^{1/2+\epsilon})$ for $\epsilon > 0$.

An important estimation for associated Laguerre polynomials is due to Koepf and Schmersau [24]. These authors have shown that $|L_n^\alpha(x)| < e^{x/2} [(n+\alpha)/x]^{\alpha/2}$ for $x \in (0, 4(n+\alpha)]$ when $n+\alpha > 0$ and α is an integer ([24], Theorem 2). We then obtain as $n \rightarrow \infty$

$$\left| \sum_{m=2, (m,N)=1}^{\infty} \frac{\Lambda(m)}{m^{(k+1)/2}} B(m) L_{n-1}^1(\ln m) \right| \sim n^{1/2} \left| \sum_{m=2, (m,N)=1}^{\infty} \frac{\Lambda(m)B(m)}{m^{k/2} \ln^{1/2} m} \right|. \quad (14)$$

Of course we have the trivial bound $|B(p^\ell)| \leq |\text{tr}[T(p^\ell)]| + p^{k-1} |\text{tr}[T(p^{\ell-2})]|$.

Now suppose that $|B(m)| \leq 2g$, which seems to follow from the Ramanujan conjecture proved in Ref. [12] (Thm. 8.2) for weight 1 or for weight 2 [31]. This is because the two solutions of the quadratic equation $1 - \lambda_{g_j}(p)p^{(1-k)/2}z + \chi_j(p)z^2 = 0$ for p not dividing N_j are of absolute value one. Then we specifically have

$$\begin{aligned} & \left| \sum_{m=2, (m,N)=1}^{\infty} \frac{\Lambda(m)}{m^{(k+1)/2}} B(m) \right| \leq \left| \sum_{m=2}^{\infty} \frac{\Lambda(m)}{m^{3/2}} B(m) \right| \\ & \leq 2g \left[\sum_{m=2}^{\infty} \frac{|\Lambda(m) - 1|}{m^{3/2}} + \zeta(3/2) - 1 \right] \leq 2g[2(\gamma - 1) + \zeta(3/2)]. \quad (15) \end{aligned}$$

This inequality together with our results above would show that for all sufficiently large n , $\tau_N(n) \geq 0$. Indeed, if $|B(m)| \leq 2g$ holds, then on the basis of combining previous results, we expect that under a condition of the form $n \geq 2\pi e^{3(\gamma-1)} e^{2\zeta(3/2)} \mathfrak{f}_N^{1/2g}$, all $\tau_N(n)$ are guaranteed to be nonnegative.

Discussion of the logarithmic derivative of L_N

In this section we provide an independent derivation of Eqs. (4) and (5). This permits us to identify the function $B(m)$, and, at least formally, the logarithmic derivative of the Hecke L -function L_N . We exhibit explicitly the polygamma constants present in Eq. (4) or (5).

We suppose that

$$\tau_N(n) = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[s^{n-1} \ln \xi_N(s) \right]_{s=1}, \tag{16}$$

which gives

$$\tau_N(n) = n \left[\frac{d}{ds} \ln \xi_N(s) \right]_{s=1} + \sum_{m=2}^n \binom{n}{m} \frac{1}{(m-1)!} \left[\frac{d^m}{ds^m} \ln \xi_N(s) \right]_{s=1}, \tag{17}$$

and show that these equations lead to Eq. (4) or (5). Based upon expressions in Ref. [28], we may write

$$\xi_N(s) = \prod_{j=1}^g \xi_{g_j}(s) = A^{s/2} N^{gs/2} (2\pi)^{-gs} \Gamma^g \left(s + \frac{k-1}{2} \right) \prod_{j=1}^g L_{g_j} \left(s + \frac{k-1}{2} \right), \tag{18}$$

where $A \equiv N^{-g} \prod_{j=1}^g N_j$. Equation (18) contains Archimedean factors of $(2\pi)^{-s} \Gamma$ and a product over local factors L_{g_j} . For future use, we note that [2]

$$(AN^g)^{s/2} \equiv \prod_{j=1}^g N_j^{s/2} = \left[N^{\nu_N} \prod_{f|m, m|N} m^{\nu_m d(N/m)} \right]^{s/2}. \tag{19}$$

We have

$$\ln \xi_N(s) = \frac{s}{2} \ln A + \frac{g}{2} s \ln N - gs \ln(2\pi) + g \ln \Gamma \left(s + \frac{k-1}{2} \right) + \sum_{j=1}^g \ln L_{g_j} \left(s + \frac{k-1}{2} \right),$$

$\text{Re } s \geq 1, \text{ for } \text{Im } s \neq 0,$ (20)

and therefore

$$\frac{\xi'_N(s)}{\xi_N(s)} = \frac{1}{2} \ln A + \frac{g}{2} \ln N - g \ln(2\pi) + g \psi \left(s + \frac{k-1}{2} \right) - \sum_{j=1}^g \sum_{\ell=1, (\ell, N)=1}^{\infty} \frac{\Lambda(\ell) b_{g_j}(\ell)}{\ell^{s+(k-1)/2}}, \tag{21}$$

where, for prime numbers p , $b_f(p^m) = \lambda^m(p)$ if $p|N$ and $b_f(p^m) = \alpha_p^m + \beta_p^m$ if $(p, N) = 1$. Here, α_p and β_p are the two solutions of the quadratic equation $T^2 - \lambda(p)T + \chi(p)p^{k-1} = 0$ [28, 31]. The form of the logarithmic derivative of the local factors in Eq. (21) follows from their Euler product and the factorization $1 - \lambda(p)X + \chi(p)p^{k-1}X^2 = (1 - \alpha_p X)(1 - \beta_p X)$, so that $\lambda(p) = \alpha_p + \beta_p$. We then have for $m \geq 2$,

$$\frac{d^m}{ds^m} \ln \xi_N(s) = g \psi^{(m-1)} \left(s + \frac{k-1}{2} \right) - (-1)^{m-1} \sum_{j=1}^g \sum_{\ell=1, (\ell, N)=1}^{\infty} \frac{\Lambda(\ell) b_{g_j}(\ell)}{\ell^{s+(k-1)/2}} \ln^{m-1} \ell, \tag{22}$$

where $\psi^{(j)}$ is the polygamma function [1, 14].

By inserting Eqs. (21) and (22) with $s \rightarrow 1^+$ into Eq. (17), using definition (3), relation (19), and simplifying, we obtain

$$\begin{aligned} \tau_N(n) = & \frac{n}{2} \ln f_N - ng \ln(2\pi) + g \sum_{m=1}^n \binom{n}{m} \frac{1}{(m-1)!} \psi^{(m-1)} \left(\frac{k+1}{2} \right) \\ & - \sum_{m=1}^n \binom{n}{m} \frac{(-1)^{m-1}}{(m-1)!} \sum_{\ell=2,(\ell,N)=1}^{\infty} \frac{\Lambda(\ell)B(\ell)}{\ell^{(k+1)/2}} \ln^{m-1} \ell, \end{aligned} \tag{23}$$

where we have identified

$$B(\ell) \equiv \sum_{j=1}^g b_{g_j}(\ell). \tag{24}$$

Instead, we may write Eq. (23) as

$$\begin{aligned} \tau_N(n) = & \frac{n}{2} \ln f_N - ng \ln(2\pi) + g \sum_{m=1}^n \binom{n}{m} \frac{1}{(m-1)!} \psi^{(m-1)} \left(\frac{k+1}{2} \right) \\ & - \sum_{\ell=2,(\ell,N)=1}^{\infty} \frac{\Lambda(\ell)B(\ell)}{\ell^{(k+1)/2}} L_{n-1}^1(\ln \ell). \end{aligned} \tag{25}$$

In short, we have from Eqs. (1) and (18) that

$$L_N \left(s + \frac{k-1}{2} \right) = A^{s/2} \prod_{j=1}^g L_{g_j} \left(s + \frac{k-1}{2} \right), \tag{26}$$

and have found that

$$\frac{d}{ds} \ln L_N \left(s + \frac{k-1}{2} \right) = \frac{1}{2} \ln A - \sum_{\ell=2,(\ell,N)=1}^{\infty} \frac{\Lambda(\ell)B(\ell)}{\ell^{s+(k-1)/2}}. \tag{27}$$

Equations (23) and (25) are in complete agreement with Eqs. (4) and (5) in view of Eq. (7) and [14]

$$g \sum_{m=1}^n \binom{n}{m} \frac{1}{(m-1)!} \psi^{(m-1)} \left(\frac{k+1}{2} \right) = g \sum_{m=1}^n (-1)^m \binom{n}{m} \sum_{j=1}^{\infty} \frac{1}{(j + \frac{k-1}{2})^m}. \tag{28}$$

The special case of the Riemann zeta function is essentially the degeneration of $N \rightarrow 1$, $\ln A \rightarrow 0$, $g \rightarrow 1$, $k \rightarrow 1$, and $B \rightarrow 1$.

Summary and Brief Discussion

We have shown that the apparent leading behaviour of the Li constants $\tau_N(n)$ is given by $O(n \ln n)$, which extends our previous work [9] on the constants λ_k corresponding to the Riemann xi function or classical *L* functions.

The dominating terms in Eqs. (4), (5), (23), or (25) are those coming from summation over polygamma constants. I.e., the dominant terms arise from the Archimedean factors in $\xi_N(s)$. Our approach, already applied to Dirichlet L-functions and Hecke L-functions of weight 2 [9], extends to a variety of generalized xi and L-functions.

The identification of the special polynomials in Eq. (5) is important for several reasons—we can bring to bear the asymptotics, bounds, orthogonality, and other properties of the associated Laguerre polynomials. For instance, the orthogonality property on the half line translates to the statement $\int_0^1 L_{n-1}^1(-\ln u)L_{m-1}^1(-\ln u)(-\ln u)du = \int_0^1 L_{n-1}^1(-\ln u)L_{m-1}^1(-\ln u)|\ln u|du = n\delta_{nm}$, where δ_{ij} is the Kronecker delta, and this seems to be connected to the possible positivity of the Weil inner product on a suitably defined space of test functions [6]. We have recently shown how the framework of the Li/Keiper constants may be written in terms of the Laguerre calculus [10].

The coefficient function $B(\ell)$ enters the logarithmic derivative of the Hecke L-function along with the von Mangoldt function Λ , as in Eq. (27). We have also shown that suitable bounds on the function B will lead to verification of a generalized Riemann hypothesis, since then the nonnegativity of the constants $\tau_N(n)$ is assured. For rank 1 or 2 L functions, the results of Deligne and Shimura [12, 31] may present the bound $|B(m)| \leq \sum_{j=1}^g |b_{g_j}(m)| \leq 2g$. We have briefly herein explored the consequence of this inequality. More broadly, proof of the generalized Ramanujan conjectures [20] may permit verification of the generalized Riemann hypothesis for automorphic L-functions on GL_m [30].

Appendix: On the integral of Eq. (11)

Here we discuss the integral

$$\begin{aligned} I_n(k) &\equiv \int_1^{(k+1)/2} \frac{(x-1)^n}{x^n} dx = \left(\frac{k-1}{2}\right)^{n+1} \frac{1}{(n+1)} {}_2F_1\left(n, n+1; n+2; \frac{1-k}{2}\right) \\ &= \frac{(k^2-1)}{4(n+1)} \left(\frac{k-1}{k+1}\right)^n {}_2F_1\left(1, 2; n+2; \frac{1-k}{2}\right), \end{aligned} \quad (\text{A.1})$$

from the hypergeometric function point of view. In this equation, the second line follows from the first from a standard transformation formula (e.g., [14], p. 1043). Although there are many other ways in which to evaluate this integral, these results are particularly suited to large values of the positive integer n . First we have, by differentiating Eq. (A.1) with respect to k and using Leibniz' rule,

$$\frac{1}{2} \left(\frac{k-1}{k+1}\right)^n = \frac{(n+1)}{(k-1)} I_n(k) - \frac{n}{(k-1)} I_{n+1}(k), \quad n \geq 0. \quad (\text{A.2})$$

Rewriting this recursion relation immediately recovers Eq. (12) derived by integration by parts. We also note that when $k = 1$, Eq. (12) or (A.2) reduces to $I_n(k) = nI_{n-1}(k)/(n - 1)$ and $I_1(1) = 0$, as it should.

We next show that for $n \geq 1$ a positive integer,

Proposition

$$\begin{aligned}
 & {}_2F_1(n, n + 1; n + 2; 1 - z) \\
 &= \frac{(n + 1)}{(n - 1)!} \left\{ \frac{(n - 2)!}{z^{n-1}} - 2 \frac{(n - 3)!}{z^{n-2}} + 6 \frac{(n - 4)!}{z^{n-3}} - \dots + (-1)^n \frac{(n - 1)!}{z} \right. \\
 &\quad \left. + \frac{(-1)^n n!}{(1 - z)^{n+1}} \left[\ln z + \sum_{j=1}^n \frac{1}{j} (1 - z)^j \right] \right\}. \tag{A.3}
 \end{aligned}$$

Equation (A.3) can be verified by mathematical induction, using

$$\frac{d}{dz} {}_2F_1(n, n + 1; n + 2; 1 - z) = -\frac{n(n + 1)}{(n + 2)} {}_2F_1(n + 1, n + 2; n + 3; 1 - z), \tag{A.4}$$

and noting that ${}_2F_1(1, 2; 3; z) = -2[z + \ln(1 - z)]/z^2$. Equation (A.3) also follows directly from

$$\begin{aligned}
 {}_2F_1(n, n + 1; n + 2; 1 - z) &= \frac{(n + 1)}{(n - 1)!} \sum_{j=0}^{\infty} \frac{(j + n - 1)!}{j!} \frac{(1 - z)^j}{(j + n + 1)} \\
 &= \frac{(n + 1)}{(n - 1)!} \sum_{j=0}^{\infty} \frac{(j + n - 1)}{(j + n + 1)} (j + n - 2) \cdots (j + 1) (1 - z)^j \\
 &= \frac{(n + 1)}{(n - 1)!} \sum_{j=0}^{\infty} \left(1 - \frac{2}{j + n + 1} \right) (j + n - 2)(j + n - 3) \cdots (j + 1) (1 - z)^j \\
 &= \frac{(n + 1)}{(n - 1)!} \left[\frac{(n - 2)!}{z^{n-1}} - 2 \sum_{j=0}^{\infty} \frac{(j + n - 2)}{(j + n + 1)} (j + n - 3)(j + n - 4) \cdots (j + 1) (1 - z)^j \right]. \tag{A.5}
 \end{aligned}$$

One proceeds in this manner, using [9]

$$\sum_{k=0}^{\infty} (k + j + 1)(k + j)(k + j - 1) \cdots (k + 1) (1 - q)^k = \frac{(j + 1)!}{q^{j+2}}, \quad |1 - q| < 1. \tag{A.6}$$

Once the factor is reached in the summand of $(j + 1)/(j + n + 1) = 1 - n/(j + n + 1)$, the summation index is shifted in the last sum, $j \rightarrow j - n - 1$, and the infinite series for $\ln z$ is used.

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