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On the Eigenstructure Assignment of Delay-Differential Systems

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Abstract

A standard canonical form is established for a class of neutral delay-differential systems. It is shown that strict system equivalence provides the connection between a given strongly controllable polynomial system matrix and the resulting canonical form. Using this canonical form, an algorithm is given for the eigenstructure assignment of a class of neutral delay-differential systems.

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1 Introduction

Consider the class of linear neutral delay differential system of the form:

$$\begin{aligned} \sum_{i=0}^{p} E_i \dot{x}(t-ih) &= \sum_{i=0}^{p} A_i x(t-ih) + \sum_{j=0}^{q} B_j u(t-jh) \\ y(t) &= \sum_{k=0}^{r} C_k x(t-kh) + \sum_{n=0}^{v} D_n u(t-nh) \end{aligned} \tag{1}$$

where $x(t)$ is an n-vector of state variables, $u(t)$ is an l-vector of controlled variables, $y(t)$ is a p-vector of observed variables, and h is a positive real constant.

The system of differential-difference equations (1) may be rewritten as a so-called generalized linear system over $\mathbb{R}[z]$:

$$\begin{bmatrix} sE(z) - A(z) & B(z) \\ -C(z) & D(z) \end{bmatrix} \begin{bmatrix} x(t) \\ -u(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -y(t) \end{bmatrix} \tag{2}$$

where $E(z), A(z), B(z)$ and $C(z)$ are respectively $n \times n, n \times n, n \times l$, and $p \times n$ matrices over $\mathbb{R}[z]$, $s = \dfrac{d}{dt}$ denotes a differential operator and z a backward shift operator i.e., $zx(t) = x(t-h)$.

Neutral delay-differential equations of the type (1) may arise in the study of lumped parameter networks interconnected by transmission lines (see for example Byrnes *et al.* [1]). The main motivation behind studying equations of the form (1) in the context of the theory of linear systems over commutative rings is the desire for a unified algebraic treatment of retarded, neutral and constant coefficient systems.

It is assumed in equations (2) that $E(z)$ is atomic at zero ie. $|E(0)| \neq 0$. This is necessary in general to guarantee causality. The system matrix $P(s,z)$ corresponding to (2) is the $(n+p) \times (n+l)$ polynomial matrix:

$$P(s,z) = \begin{bmatrix} sE(z) - A(z) & B(z) \\ -C(z) & D(z) \end{bmatrix} \tag{3}$$

The transfer function of (2), or equivalently of (3) is the $p \times l$ rational matrix given by:

$$G(s,z) = C(z)\left[sE(z) - A(z)\right]^{-1} B(z) + D(z) \tag{4}$$

In his pioneering work, Rosenbrock [7] introduced the concept of strict-system-equivalence (SSE) for system matrices over $\mathbb{R}[s]$ describing ordinary differential/difference systems. This concept is extended for system matrices of the type (3) as follows.

Definition 1.1 *Two system matrices $P_1(s,z)$ and $P_2(s,z)$ of the form (3) having the same size are SSE if there they are related by:*

$$\begin{bmatrix} M(s,z) & 0 \\ X(s,z) & I_p \end{bmatrix} \underbrace{\begin{bmatrix} sE_1(z) - A_1(z) & B_1(z) \\ -C_1(z) & D_1(z) \end{bmatrix}}_{P_1(s,z)} = \underbrace{\begin{bmatrix} sE_2(z) - A_2(z) & B_2(z) \\ -C_2(z) & D_2(z) \end{bmatrix}}_{P_2(s,z)} \begin{bmatrix} N(s,z) & Y(s,z) \\ 0 & I_l \end{bmatrix} \tag{5}$$

where $M(s,z)$ and $N(s,z)$ are $n \times n$ unimodular matrices over $\mathbb{R}[s,z]$ and $X(s,z), Y(s,z)$ are polynomial matrices of appropriate dimensions.

Following the terminology of Rosenbrock, Spong [9] introduced the concepts of restricted-system-equivalence (RSE) and weak-restricted-system-equivalence (WRSE) for system matrices of the type (3). These are defined as follows:

Definition 1.2 *Two system matrices $P_1(s,z)$ and $P_2(s,z)$ of the form (3) are RSE (WRSE) if there exist unimodular $n \times n$ matrices $M(z)$ and $N(z)$ over $\mathbb{R}[z](\mathbb{R}(z))$ such that:*

$$\begin{bmatrix} M(z) & 0 \\ 0 & I_p \end{bmatrix} \underbrace{\begin{bmatrix} sE_1(z) - A_1(z) & B_1(z) \\ -C_1(z) & D_1(z) \end{bmatrix}}_{P_1(s,z)} = \underbrace{\begin{bmatrix} sE_2(z) - A_2(z) & B_2(z) \\ -C_2(z) & D_2(z) \end{bmatrix}}_{P_2(s,z)} \begin{bmatrix} N(z) & 0 \\ 0 & I_l \end{bmatrix} \tag{6}$$

Clearly SSE $\implies$ RSE $\implies$ WRSE.

Both RSE and WRSE preserve the form of the system matrix , the system order and the transfer function of the system (1). Zerz [10] pointed out that the controllability and observability of a system (1) is closely related to its zero structure. The zero structure of a polynomial system matrix (3) is completely captured by the determinantal ideals as defined by the following.

Definition 1.3 *Let $P(s,z) \in \mathbb{R}^{r_1 \times r_2}[s,z]$, the ith order determinantal ideal $\mathcal{I}_i^{[P]}$ of the polynomial matrix $P(s,z)$ is the ideal generated by the ith order minors of $P(s,z)$.*

The determinantal ideals $\mathcal{I}_i^{[P]}$ of $P(s,z)$ satisfy the following inclusion

$$\mathbb{R}[s,z] \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \dots \mathcal{I}_\mu \tag{7}$$

where μ is the normal rank of $P(s,z)$.

Lemma 1.4 [6]
Suppose that two polynomial matrices $P(s,z)$ and $Q(s,z) \in \mathbb{R}^{r_1 \times r_2}[s,z]$, are related by SSE and let $\mathcal{I}_j^{[P]}$, $\mathcal{I}_j^{[Q]}$, $\quad j = 1, \dots, h = \min(r_1, r_2)$ denote the determinantal ideal of order j generated by the $j \times j$ minors of $P(s,z)$ and $Q(s,z)$ respectively. Then

$$\mathcal{I}_i^{[P]} = \mathcal{I}_i^{[Q]}, \forall i = 0, \dots, h \tag{8}$$

Lemma 1.5 [4]
The transformation of SSE given in (5) preserves the transfer function of $P(s,z)$ and the determinantal ideals of the matrices:

$$sE(z) - A(z), P(s,z), \begin{bmatrix} sE(z) - A(z) & B(z) \end{bmatrix}, \begin{bmatrix} sE(z) - A(z) \\ -C(z) \end{bmatrix} \tag{9}$$

Using the definitions given by Zerz [10], the system (1) is said to be strongly controllable if the matrix:

$$\mathcal{C} = \left[\begin{array}{cc} sE(z) - A(z) & B(z) \end{array} \right] \tag{10}$$

has full rank for all $(s, z) \in \mathbb{C}^2$ and canonical if furthermore the matrix:

$$\mathcal{O} = \left[\begin{array}{c} sE(z) - A(z) \\ -C(z) \end{array} \right] \tag{11}$$

has no non-trivial factors in $\mathbb{R}[s, z]$.

2 Reduction to standard canonical form

In what follows, we will consider the problem of realization for linear neutral delay-differential systems. Our approach is influenced by results obtained in the context of systems over a commutative ring. For background on the problem of realization of linear systems over commutative rings, see for example, Kamen [5], Rouchaleau and Sontag [8], and Eising and Hautus [2]. Spong [9] has given some results on the realization of neutral systems from an algebraic approach. He used the concept of WRSE to obtain a two-step realization procedure for the construction of canonical neutral realizations for a large class of transfer functions. In the following, we present a direct method for the canonical realization of a large class of neutral delay-differential transfer functions. Although the SISO case is used, the method can readily be extended to the MIMO case.

Theorem 2.1 *Let $P(s, z)$ be an $(n + 1) \times (n + 1)$ canonical system matrix in the form (3) with $B(z)$ having no zeros. Also, let*

$$|sE(z) - A(z)| = \sum_{i=0}^{n} e_i(z)s^{n-i}, \quad (e_0(z) \text{ monic}) \tag{12}$$

and the transfer function of $P(s, z)$ be given by:

$$g(s, z) = \frac{n(s, z)}{d(s, z)} + r(z) \tag{13}$$

where,

$$\begin{array}{rcl} d(s, z) & = & e_0(z)s^n + e_1(z)s^{n-1} + \ldots + e_n(z), \\ n(s, z) & = & c_1(z) + c_2(z)s + \ldots + c_n(z)s^{n-1} \end{array} \tag{14}$$

have no common non-trivial factors in $\mathbb{R}[s,z]$. *Then,* $P(s,z)$ *is SSE to the standard canonical form:*

$$\bar{P}(s,z) = \begin{bmatrix} s\bar{E}(z) - F(z) & E_n \\ -\bar{c}(z) & r(z) \end{bmatrix} \tag{15}$$

where

$$\bar{E}(z) = \begin{bmatrix} I_{n-1} & 0 \\ 0 & e_0(z) \end{bmatrix}, \tag{16}$$

$F(z)$ *is the* $n \times n$ *companion matrix:*

$$F(z) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -e_n(z) & -e_{n-1}(z) & -e_{n-2}(z) & \cdots & -e_1(z) \end{bmatrix} \tag{17}$$

$$\bar{c}(z) = \begin{bmatrix} c_1(z) & c_2(z) & \cdots & c_n(z) \end{bmatrix}, r(z) = D(z). \tag{18}$$

and E_n *is the* n*th column of the identity matrix* I_n.

Proof. It can be easily verified that $P(s,z)$ and $\bar{P}(s,z)$ both are canonical and give rise to the same transfer function $g(s,z)$. Therefore by a result given by Frost and Boudellioua [3], $P(s,z)$ and $\bar{P}(s,z)$ are SSE since both can be reduced by SSE to the same polynomial system matrix:

$$S(s,z) = \left[\begin{array}{cc|c} I_{n-1} & 0 & 0 \\ 0 & d(s,z) & 1 \\ \hline 0 & -n(s,z) & 0 \end{array}\right] \tag{19}$$

■

Example 2.2 *Consider the system matrix:*

$$P(s,z) = \left[\begin{array}{cc|c} s & 0 & 1 \\ 0 & s(z+1)+1 & -1 \\ \hline -1 & -z & 0 \end{array}\right] \tag{20}$$

It can be easily verified that $P(s,z)$ *is canonical and that the transfer function of* $P(s,z)$ *is:*

$$g(s,z) = \frac{s+1}{s^2(z+1)+s} \tag{21}$$

By virtue of Theorem 2.1, $P(s,z)$ is SSE to the standard canonical system matrix:

$$\bar{P}(s,z) = \left[\begin{array}{cc|c} s & -1 & 0 \\ 0 & s(z+1)+1 & 1 \\ \hline -1 & -1 & 0 \end{array}\right] \tag{22}$$

as both $P(s,z)$ and $\bar{P}(s,z)$ are SSE to the system matrix:

$$S(s,z) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & s^2(z+1)+s & 1 \\ \hline 0 & -s-1 & 0 \end{array}\right] \tag{23}$$

The transformations that reduce $P(s,z)$ to $\bar{P}(s,z)$ is one of SSE in which,

$$\begin{bmatrix} M(s,z) & 0 \\ X(s,z) & 1 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{24}$$

and

$$\begin{bmatrix} N(s,z) & Y(s,z) \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & z+1 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{25}$$

It should be pointed out that despite the fact that $P(s,z)$ and $\bar{P}(s,z)$ have the same form (3), the transformation of SSE between $P(s,z)$ and $\bar{P}(s,z)$ is over $\mathbb{R}[s,z]$ and cannot be in general replaced by a transformation involving only matrices over $\mathbb{R}[z]$.

3 Eigenstructure Assignment

Having established a standard canonical form (15) for system matrices in state-space form (3), we now discuss the usefulness of this canonical form in the eigenstructure assignment problem.

Byrnes *et al.* [1] presented a solution to the problem of feedback stabilization of neutral delay differential systems using an associated Riccati equation. However, to our knowledge the problem of eigenstructure assignment for neutral systems has not been studied so far. In the following we present a sufficient condition for the eigenstructure of a SISO system matrix (3) to be arbitrarily assigned using polynomial state feedback.

Let $P(s,z)$ be a $(n+1)\times(n+1)$ system matrix over $\mathbb{R}[s,z]$ in the form (4.35). Then the closed-loop system matrix is given by

$$P_c(s,z) = \begin{bmatrix} sE(z) - A(z) + B(z)K(s,z) & B(z) \\ -C(z) & 0 \end{bmatrix} \tag{26}$$

where $K(s,z)$ is a $n \times n$ state feedback matrix over $\mathbb{R}[s,z]$.
Now, it can be shown that:

$$|sE(z) - A(z)| = \sum_{i=0}^{n} e_i(z) s^{n-i} \tag{27}$$

where $e_0(z)$ is monic and $e_i(z) \in \mathbb{R}[s,z], i = 0, 1, \ldots, n$. The system matrix $P(s,z)$ is said to be eigenstructure-assignable over $\mathbb{R}[s,z]$ if for any $\lambda_1(z), \lambda_2(z), \ldots, \lambda_n(z)$ belonging to $\mathbb{R}[z]$ there exists a matrix $K(s,z)$ over $\mathbb{R}[s,z]$ such that

$$|sE(z) - A(z) + B(z)K(s,z)| = \prod_{i=1}^{n} e_0(z)\,[s - \lambda_i(z)] \tag{28}$$

Theorem 3.1 *If the system matrix $P(s,z)$ is strongly controllable then it is eigenstructure assignable.*

Proof. Since $P(s,z)$ is strongly controllable, there exist $n \times n$ unimodular matrices $M(s,z)$ and $N(s,z)$ over $\mathbb{R}[s,z]$ such that

$$\begin{aligned} M(s,z) \begin{bmatrix} sE(z) - A(z) + B(z)K(s,z) & B(z) \end{bmatrix} N(s,z) \\ = \begin{bmatrix} s\bar{E}(z) - F(z) + E_n K'(z) & E_n \end{bmatrix} \end{aligned} \tag{29}$$

where $K'(z) = K(s,z)N(s,z)$ and $s\bar{E}(z) - F(z)$ has the canonical form (15).

Let the feedback matrix

$$K'(z) = \begin{bmatrix} k'_n(z) & k'_{n-1}(z) & \cdots & k'_1(z) \end{bmatrix} \tag{30}$$

and the last row of $s\bar{E}(z) - F(z)$ be given by

$$\begin{bmatrix} e_n(z) & e_{n-1}(z) & \cdots & e_1(z) \end{bmatrix} \tag{31}$$

Then the matrix $s\bar{E}(z) - F(z) + E_n K'(z)$ has the same canonical form as $s\bar{E}(z) - F(z)$ except for the last row

$$\begin{bmatrix} e_n(z) + k'_n(z) & e_{n-1}(z) + k'_{n-1}(z) & \cdots & e_1(z) + k'_1(z) \end{bmatrix} \tag{32}$$

Since

$$K'(z) = K(s,z)N(s,z), \tag{33}$$

it follows that the desired feedback matrix

$$K(s,z) = K'(z)N^{-1}(s,z) \tag{34}$$

Let $k'_i(z) = e_0(z)\bar{k}'_i(z) - e_i(z), \quad (i = 1, 2, \ldots, n)$.
Then the last row of $s\bar{E}(z) - F(z) + E_n K'(z)$ becomes

$$\begin{bmatrix} e_0(z)\bar{k}'_n(z) & e_0(z)\bar{k}'_{n-1}(z) & \cdots e_0(z)\left[s + \bar{k}'_1(z)\right] \end{bmatrix} \tag{35}$$

so that $\bar{k}'_i(z), (i = 1, 2, \ldots, n)$ is obtained by equating the coefficients of s in

$$s^n + \bar{k}'_1(z)s^{n-1} + \ldots + \bar{k}'_1(z) = \prod_{i=0}^{n} [s - \lambda_i(z)] \tag{36}$$

■

Example 3.2 *Consider the SISO polynomial system matrix $P(s,z)$ given in Example 2.2, i.e.,*

$$P(s,z)=\left[\begin{array}{cc|c} s & 0 & 1 \\ 0 & s(z+1)+1 & -1 \\ \hline -1 & -z & 0 \end{array}\right] \tag{37}$$

The characteristic polynomial associated with $P(s,z)$ is

$$p(s,z)=|sE(z)-A(z)|=s^2(z+1)+s \tag{38}$$

Suppose we wish the poles of the closed loop system to be -1 and -2 so that the closed loop characteristic polynomial is

$$p_c(s,z)=(z+1)(s^2+3s+2) \tag{39}$$

It follows that

$$\bar{k}'_1(z)=3, \quad \bar{k}'_2(z)=2 \tag{40}$$

Using the matrix $N(s,z)$ in Example 2.2,

$$K(z):=K'(z)N^{-1}(s,z)=\left[\begin{array}{cc} 2(z+1) & 2z(z+1) \end{array}\right] \tag{41}$$

It is easy to verify that the closed loop system matrix

$$P_c(s,z)=\left[\begin{array}{cc|c} s+2(z+1) & 2z(z+1) & 1 \\ -2(z+1) & s(z+1)-(z+1)(2z-1) & -1 \\ \hline -1 & -z & 0 \end{array}\right] \tag{42}$$

does have the desired eigenstructure.

4 Conclusions

In this paper a standard canonical form is established for polynomial system matrices arising from a class of neutral delay-differential systems. The standard form is canonical in the sense given by Zerz [10]. Using this canonical form, conditions are given under which a system matrix is eigenstructure assignable. An algorithm is given for the eigenstructure assignment of the system.

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