Semicontinuous Representability of Interval Orders on a Metrizable Topological Space

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Abstract
An interval order $\succeq$ on an arbitrary set $X$ is said to be representable by a pair $(u, v)$ of real-valued functions on $X$ if $x \succeq y$ is equivalent to $u(x) \leq v(y)$ for all $x, y \in X$. We say that a topology $\tau$ on $X$ is completely useful in the sense of an interval order (i.o. completely useful for the sake of brevity) if every upper semicontinuous interval order $\succeq$ on the topological space $(X, \tau)$ is representable by a pair $(u, v)$ of real-valued functions with $v$ upper semicontinuous. We show that every i.o. completely useful topology on $X$ is also completely useful (i.e., every upper semicontinuous total preorder $\preceq$ on $(X, \tau)$ is representable by an upper semicontinuous utility function). Further we show that a metrizable topology $\tau$ on $X$ is i.o. completely useful if and only if $\tau$ is separable.

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1 Introduction

Recall that a total preorder \( \preceq \) on a topological space \((X, \tau)\) is said to be upper (lower) semicontinuous if \( L_\prec(x) = \{z \in X : z \prec x\}\) (respectively, \( U_\prec(x) = \{z \in X : x \prec z\}\)) is an open subset of \(X\) for every \(x \in X\) (we have denoted by \(\prec\) the strict part of \(\preceq\)). Further, \(\preceq\) is said to be continuous if it is both upper and lower semicontinuous.

Herden [9] introduced the concept of a useful topology as a topology \(\tau\) on an arbitrary set \(X\) such that every continuous total preorder \(\preceq\) on the topological space \((X, \tau)\) is representable by means of a continuous utility function (see also Herden and Pallack [10]).

More recently, Bosi and Herden [1] characterized completely useful topologies on a set \(X\), where a topology \(\tau\) on \(X\) is said to be completely useful if every upper semicontinuous total preorder \(\preceq\) on \((X, \tau)\) is representable by an upper semicontinuous utility function. In this paper we introduce the notion of a completely useful topology in the sense of an interval order (i.o. completely useful topology for the sake of brevity) as a topology \(\tau\) on a set \(X\) such that every upper semicontinuous interval order \(\preceq\) on the topological space \((X, \tau)\) is representable by a pair \((u, v)\) of real-valued functions with \(v\) upper semicontinuous. Recall that an interval order \(\preceq\) on \(X\) is said to be representable by a pair \((u, v)\) of real-valued functions on \(X\) if \(x \preceq y\) is equivalent to \(u(x) \leq v(y)\) for all \(x, y \in X\).

In this paper we show that every i.o. completely useful topology is completely useful. Further we show that a metrizable topology \(\tau\) on \(X\) is i.o. completely useful if and only if \(\tau\) is separable, and therefore that the concepts of usefulness, complete usefulness and i.o. complete usefulness are equivalent when we consider metrizable topologies (see the Theorem in Estévez and Hervés [7] and Corollary 4.5 in Bosi and Herden [1]).

In order to justify the introduction of the concept of an i.o. completely useful topology, let us consider the following example of a completely useful topology which fails to be i.o. completely useful.

Example 1.1 Let \(X\) be the real line and consider the topology \(\tau\) on \(X\) which is induced by the sets \([-\infty, r]\) where \(r\) runs through all reals. It has been already shown that \(\tau\) is a completely useful topology (see Bosi and Herden [1, Theorem 3.2 and Example 3.5]). Let \(\preceq\) be the interval order on \(X\) which is defined by \(x \preceq y \iff x < y + 1\) \((x, y \in X)\). Then \(\preceq\) is not i.o. separable (see Bosi et al. [3, Remark 4 (c)]) and therefore \(\preceq\) is not representable by means of a pair of real-valued functions (see Bosi et al. [3, Theorem 1]). In order to prove that the topology \(\tau\) on \(X\) is not i.o. completely useful it remains to show that the interval order \(\preceq\) is upper semicontinuous. Consider the fact that \(L_\prec(x) = \{z \in X : z \prec x\}\) is equal to the set \(\{z \in X : z + 1 \leq x\}\) for every \(x \in X\). Hence if \(z\) is any element of \(L_\prec(x)\) we have that \([-\infty, z]\) is an open
subset of $X$ which contains $z$ and is contained in $L_{\prec}(x)$. This consideration finishes this example.

2 Notation and preliminaries

In this paper we are concerned with the existence of a real representation $(u, v)$ of an interval order $\preceq$ on a topological space $(X, \tau)$ such that $v$ is upper semicontinuous. More precisely, we want to characterize the topologies $\tau$ on a set $X$ which are metrizable and such that every upper semicontinuous interval order $\preceq$ on $(X, \tau)$ is representable by a pair $(u, v)$ of real-valued functions with $v$ upper semicontinuous.

An interval order $\preceq$ on a set $X$ is a binary relation on $X$ which is reflexive and satisfies the following condition for all $x, y, z, w \in X$:

\[(x \preceq z) \land (y \preceq w) \Rightarrow (x \preceq w) \lor (y \preceq z).\]

It is well known that interval orders are of interest in economics and social sciences since they are total binary relations which are not necessarily transitive (see Oloriz, Candeal and Indurain [11]). It is clear that a total preorder (i.e. a reflexive, transitive and total binary relation) $\preceq$ on a set $X$ is an interval order. The strict part of an interval order $\preceq$ on a set $X$ will be denoted by $\prec$.

It is well known that if $\preceq$ is an interval order on a set $X$ and we define, for every element $x \in X$, $L_{\prec}(x) = \{ z \in X : z < x \}$, $U_{\prec}(x) = \{ z \in X : x < z \}$, then $L_{\prec}(X) = \{ L_{\prec}(x) : x \in X \}$ and $U_{\prec}(X) = \{ U_{\prec}(x) : x \in X \}$ are both linearly ordered by set inclusion (see e.g. Rabinovitch [12, Theorem 2]).

Fishburn [8] proved that if $\preceq$ is an interval order on a set $X$, then the following two binary relations $\preceq^*$ and $\preceq^{**}$ on $X$ are both total preorders:

\[x \preceq^* y \Leftrightarrow (z \preceq x \Rightarrow z \preceq y) \ \forall \ z \in X,\]

\[x \preceq^{**} y \Leftrightarrow (y \preceq z \Rightarrow x \preceq z) \ \forall \ z \in X.\]

The strict parts of $\preceq^*$ and $\preceq^{**}$ will be denoted by $\prec^*$ and respectively $\prec^{**}$. In the particular case when $\preceq$ is a total preorder it happens that $\preceq = \preceq^* = \preceq^{**}$.

We recall that a real-valued function $u$ is a utility function for a total preorder $\preceq$ on a set $X$ if, for all $x, y \in X$,

\[x \preceq y \Leftrightarrow u(x) \leq u(y).\]

An interval order $\preceq$ on a set $X$ is representable by means of a pair $(u, v)$ of real-valued functions on $X$ provided that, for all $x, y \in X$,

\[x \preceq y \Leftrightarrow u(x) \leq v(y).\]
An interval order \( \preceq \) on a set \( X \) is said to be i.o. separable if there exists a countable subset \( D \) of \( X \) such that for all \( x, y \in X \) with \( x \prec y \) there exists \( d \in D \) such that \( x \prec d \preceq \star \star y \).

An interval order \( \preceq \) on a topological space \((X, \tau)\) is said to be upper semicontinuous if \( L_x(x) \) is an open subset of \( X \) for every \( x \in X \).

In Bosi and Herden [1] a topology \( \tau \) on a set \( X \) is said to be completely useful if every upper semicontinuous total preorder \( \preceq \) on \( X \) is representable by an upper semicontinuous utility function.

Let us now present the main definition of this paper.

**Definition 2.1** We say that a topology \( \tau \) on a set \( X \) is completely useful in the sense of an interval order (i.o. completely useful for the sake of brevity) if every upper semicontinuous interval order \( \preceq \) on \((X, \tau)\) is representable by a pair \((u, v)\) of real-valued functions with \( v \) upper semicontinuous.

### 3 Characterization of i.o. completely useful metrizable topologies

In order to characterize i.o. completely useful topologies which are metrizable we need the following two lemmas.

**Lemma 3.1** Every i.o. completely useful topology \( \tau \) on a set \( X \) is completely useful.

*Proof.* Let \( \tau \) be an i.o. completely useful topology on a set \( X \) and consider an upper semicontinuous interval order \( \preceq \) on \((X, \tau)\). Then \( \preceq \) is an upper semicontinuous interval order which is representable by means of a pair \((u, v)\) of real-valued functions on \( X \). We know that in this case \( \preceq \) is i.o. separable (see Bosi et al. [3, Theorem 1]) and therefore, since we have that \( \preceq = \preceq \star \star \), there exists a countable subset \( D \) of \( X \) such that for all \( x, y \in X \) with \( x \prec y \) there exists \( d \in D \) such that \( x \prec d \preceq y \). Hence, \( \preceq \) is order separable in the sense of Debreu (see e.g. Bridges and Mehta [5]) and there exists an upper semicontinuous utility function on the totally preordered topological space \((X, \tau, \preceq)\) (see Debreu [6, Proposition 5]). \(\square\)

We recall that a gap of a subset \( S \) of the real line \( \mathbb{R} \) is a maximal nondegenerate interval which is disjoint from \( S \) and has a lower bound and an upper bound in \( S \) (see e.g. Bridges and Mehta [5]).

**Lemma 3.2** A second countable topology \( \tau \) on a set \( X \) is i.o. completely useful.
Proof. Let \( \tau \) be a second countable topology on a set \( X \) and consider an upper semicontinuous interval order \( \preceq \) on \( (X, \tau) \). Then \( \preceq \) is representable by a pair \((u', v')\) of real-valued functions on \( X \) (see Bridges [4, Proposition 2.3]) where \( v' \) may be chosen so that it is a utility function for the total preorder \( \preceq^* \) (see Bosi et al. [3, Theorem 1, (ix)]). From Debreu Open Gap Lemma (see e.g. Bridges and Mehta [5, Lemma 3.1.3]) there exists a strictly increasing function \( h : \mathbb{R} \rightarrow \mathbb{R} \) such that every gap of \((h \circ v')(X)\) is open. Observe that the total preorder \( \preceq^* \) is upper semicontinuous since the interval order \( \preceq \) is upper semicontinuous and for every \( x \in X \) we have that \( L \prec^* (x) = \bigcup_{\xi \in X, \xi \preceq x} L \prec (\xi) \). Hence, if we let \( u := h \circ u' \) and \( v := h \circ v' \) it is immediate to check that the pair \((u, v)\) of real-valued functions on \( X \) represents the interval order \( \preceq \) and that \( v \) is upper semicontinuous. This consideration completes the proof. \( \square \)

Remark 3.3 Lemma 3.2 may be viewed as a generalization of Rader’s Theorem (see Rader [13]) which states that there is an upper semicontinuous utility function \( u \) for every upper semicontinuous total preorder \( \preceq \) on a second countable topological space \((X, \tau)\).

The following characterization of an i.o. completely useful topology in the metrizable case is similar to Corollary 4.5 in Bosi and Herden [1] and to the Theorem in Estévez and Hervés [7].

Proposition 3.4 Let \( \tau \) be a metrizable topology on a set \( X \). Then the following conditions are equivalent:

(i) \( \tau \) is i.o. completely useful;

(ii) \( \tau \) is separable.

Proof. (i) \( \Rightarrow \) (ii). If \( \tau \) is an i.o. completely useful and metrizable topology on a set \( X \) then from Lemma 3.1 we have that \( \tau \) is completely useful and therefore it is separable from Bosi and Herden [1, Corollary 4.5].

(ii) \( \Rightarrow \) (i). If \( \tau \) is a metrizable and separable topology on a set \( X \) then \( \tau \) is second countable and therefore it is i.o. completely useful by Lemma 3.2. This consideration completes the proof. \( \square \)

References


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