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# Semicontinuous Representability of Interval Orders on a Metrizable Topological Space

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**Abstract**

An interval order $\precsim$ on an arbitrary set $X$ is said to be representable by a pair $(u, v)$ of real-valued functions on $X$ if $x \precsim y$ is equivalent to $u(x) \leq v(y)$ for all $x, y \in X$. We say that a topology $\tau$ on $X$ is *completely useful in the sense of an interval order* (*i.o. completely useful* for the sake of brevity) if every upper semicontinuous interval order $\precsim$ on the topological space $(X, \tau)$ is representable by a pair $(u, v)$ of real-valued functions with $v$ upper semicontinuous. We show that every i.o. completely useful topology on $X$ is also *completely useful* (i.e., every upper semicontinuous total preorder $\precsim$ on $(X, \tau)$ is representable by an upper semicontinuous utility function). Further we show that a metrizable topology $\tau$ on $X$ is i.o. completely useful if and only if $\tau$ is separable.

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# 1 Introduction

Recall that a total preorder $\precsim$ on a topological space $(X, \tau)$ is said to be *upper* (*lower*) *semicontinuous* if $L_{\prec}(x) = \{z \in X : z \prec x\}$ (respectively, $U_{\prec}(x) = \{z \in X : x \prec z\}$) is an open subset of $X$ for every $x \in X$ (we have denoted by $\prec$ the strict part of $\precsim$). Further, $\precsim$ is said to be *continuous* if it is both upper and lower semicontinuous.

Herden [9] introduced the concept of a *useful topology* as a topology $\tau$ on an arbitrary set $X$ such that every *continuous total preorder* $\precsim$ on the topological space $(X, \tau)$ is representable by means of a continuous utility function (see also Herden and Pallack [10]).

More recently, Bosi and Herden [1] characterized *completely useful topologies* on a set $X$, where a topology $\tau$ on $X$ is said to be *completely useful* if every upper semicontinuous total preorder $\precsim$ on $(X, \tau)$ is representable by an upper semicontinuous utility function. In this paper we introduce the notion of a *completely useful topology in the sense of an interval order* (*i.o. completely useful topology* for the sake of brevity) as a topology $\tau$ on a set $X$ such that every upper semicontinuous interval order $\precsim$ on the topological space $(X, \tau)$ is representable by a pair $(u, v)$ of real-valued functions with $v$ upper semicontinuous. Recall that an interval order $\precsim$ on $X$ is said to be representable by a pair $(u, v)$ of real-valued functions on $X$ if $x \precsim y$ is equivalent to $u(x) \leq v(y)$ for all $x, y \in X$.

In this paper we show that every i.o. completely useful topology is completely useful. Further we show that a metrizable topology $\tau$ on $X$ is i.o. completely useful if and only if $\tau$ is separable, and therefore that the concepts of usefulness, complete usefulness and i.o. complete usefulness are equivalent when we consider metrizable topologies (see the Theorem in Estévez and Hervés [7] and Corollary 4.5 in Bosi and Herden [1]).

In order to justify the introduction of the concept of an i.o. completely useful topology, let us consider the following example of a completely useful topology which fails to be i.o. completely useful.

**Example 1.1** Let $X$ be the real line and consider the topology $\tau$ on $X$ which is induced by the sets $]-\infty, r]$ where $r$ runs through all reals. It has been already shown that $\tau$ is a completely useful topology (see Bosi and Herden [1, Theorem 3.2 and Example 3.5]). Let $\precsim$ be the interval order on $X$ which is defined by $x \precsim y \Leftrightarrow x < y + 1$ $(x, y \in X)$. Then $\precsim$ is not i.o. separable (see Bosi et al. [3, Remark 4 (c)]) and therefore $\precsim$ is not representable by means of a pair of real-valued functions (see Bosi et al. [3, Theorem 1]). In order to prove that the topology $\tau$ on $X$ is not i.o. completely useful it remains to show that the interval order $\precsim$ is upper semicontinuous. Consider the fact that $L_{\prec}(x) = \{z \in X : z \prec x\}$ is equal to the set $\{z \in X : z + 1 \leq x\}$ for every $x \in X$. Hence if $z$ is any element of $L_{\prec}(x)$ we have that $]-\infty, z]$ is an open subset of $X$ which contains $z$ and is contained in $L_{\prec}(x)$. This consideration finishes this example.

## 2 Notation and preliminaries

In this paper we are concerned with the existence of a real representation $(u, v)$ of an interval order $\precsim$ on a topological space $(X, \tau)$ such that $v$ is upper semicontinuous. More precisely, we want to characterize the topologies $\tau$ on a set $X$ which are metrizable and such that every upper semicontinuous interval order $\precsim$ on $(X, \tau)$ is representable by a pair $(u, v)$ of real-valued functions with $v$ upper semicontinuous.

An *interval order* $\precsim$ on a set $X$ is a binary relation on $X$ which is reflexive and satisfies the following condition for all $x, y, z, w \in X$:

$$(x \precsim z) \wedge (y \precsim w) \Rightarrow (x \precsim w) \vee (y \precsim z).$$

It is well known that interval orders are of interest in economics and social sciences since they are total binary relations which are not necessarily transitive (see Oloriz, Candeal and Induráin [11]). It is clear that a total preorder (i.e. a reflexive, transitive and total binary relation) $\precsim$ on a set $X$ is an interval order. The strict part of an interval order $\precsim$ on a set $X$ will be denoted by $\prec$.

It is well known that if $\precsim$ is an interval order on a set $X$ and we define, for every element $x \in X$, $L_{\prec}(x) = \{z \in X : z \prec x\}$, $U_{\prec}(x) = \{z \in X : x \prec z\}$, then $L_{\prec}(X) = \{L_{\prec}(x) : x \in X\}$ and $U_{\prec}(X) = \{U_{\prec}(x) : x \in X\}$ are both linearly ordered by set inclusion (see e.g. Rabinovitch [12, Theorem 2]).

Fishburn [8] proved that if $\precsim$ is an interval order on a set $X$, then the following two binary relations $\precsim^*$ and $\precsim^{**}$ on $X$ are both total preorders:

$$x \precsim^* y \Leftrightarrow (z \precsim x \Rightarrow z \precsim y) \ \forall z \in X,$$

$$x \precsim^{**} y \Leftrightarrow (y \precsim z \Rightarrow x \precsim z) \ \forall z \in X.$$

The strict parts of $\precsim^*$ and $\precsim^{**}$ will be denoted by $\prec^*$ and respectively $\prec^{**}$. In the particular case when $\precsim$ is a total preorder it happens that $\precsim = \precsim^* = \precsim^{**}$.

We recall that a real-valued function $u$ is a *utility function* for a total preorder $\precsim$ on a set $X$ if, for all $x, y \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq u(y).$$

An interval order $\precsim$ on a set $X$ is *representable by means of a pair* $(u, v)$ of real-valued functions on $X$ provided that, for all $x, y \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq v(y).$$

An interval order $\precsim$ on a set $X$ is said to be *i.o. separable* if there exists a countable subset $D$ of $X$ such that for all $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \prec d \precsim^{**} y$.

An interval order $\precsim$ on a topological space $(X, \tau)$ is said to be *upper semicontinuous* if $L_{\prec}(x)$ is an open subset of $X$ for every $x \in X$.

In Bosi and Herden [1] a topology $\tau$ on a set $X$ is said to be *completely useful* if every upper semicontinuous total preorder $\precsim$ on $X$ is representable by an upper semicontinuous utility function.

Let us now present the main definition of this paper.

**Definition 2.1** We say that a topology $\tau$ on a set $X$ is *completely useful in the sense of an interval order* (*i.o. completely useful* for the sake of brevity) if every upper semicontinuous interval order $\precsim$ on $(X, \tau)$ is representable by a pair $(u, v)$ of real-valued functions with $v$ upper semicontinuous.

# 3 Characterization of i.o. completely useful metrizable topologies

In order to characterize i.o. completely useful topologies which are metrizable we need the following two lemmas.

**Lemma 3.1** *Every i.o. completely useful topology $\tau$ on a set $X$ is completely useful.*

*Proof.* Let $\tau$ be an i.o. completely useful topology on a set $X$ and consider an upper semicontinuous total preorder $\precsim$ on $(X, \tau)$. Then $\precsim$ is an upper semicontinuous interval order which is representable by means of a pair $(u, v)$ of real-valued functions on $X$. We know that in this case $\precsim$ is i.o. separable (see Bosi et al. [3, Theorem 1]) and therefore, since we have that $\precsim = \precsim^{**}$, there exists a countable subset $D$ of $X$ such that for all $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \prec d \precsim y$. Hence, $\precsim$ is *order separable in the sense of Debreu* (see e.g. Bridges and Mehta [5]) and there exists an upper semicontinuous utility function on the totally preordered topological space $(X, \tau, \precsim)$ (see Debreu [6, Proposition 5]). □

We recall that a gap of a subset $S$ of the real line $\mathbb{R}$ is a maximal nondegenerate interval which is disjoint from $S$ and has a lower bound and an upper bound in $S$ (see e.g. Bridges and Mehta [5]).

**Lemma 3.2** *A second countable topology $\tau$ on a set $X$ is i.o. completely useful.*

*Proof.* Let $\tau$ be a second countable topology on a set $X$ and consider an upper semicontinuous interval order $\precsim$ on $(X, \tau)$. Then $\precsim$ is representable by a pair $(u', v')$ of real-valued functions on $X$ (see Bridges [4, Proposition 2.3]) where $v'$ may be chosen so that it is a utility function for the total preorder $\precsim^*$ (see Bosi et al. [3, Theorem 1, (ix)]). From Debreu Open Gap Lemma (see e.g. Bridges and Mehta [5, Lemma 3.1.3]) there exists a strictly increasing function $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that every gap of $(h \circ v')(X)$ is open. Observe that the total preorder $\precsim^*$ is upper semicontinuous since the interval order $\precsim$ is upper semicontinuous and for every $x \in X$ we have that $L_{\prec^*}(x) = \bigcup_{\{\xi \in X, \xi \precsim x\}} L_{\prec}(\xi)$. Hence, if we let $u := h \circ u'$ and $v := h \circ v'$ it is immediate to check that the pair $(u, v)$ of real-valued functions on $X$ represents the interval order $\precsim$ and that $v$ is upper semicontinuous. This consideration completes the proof. □

**Remark 3.3** Lemma 3.2 may be viewed as a generalization of Rader's Theorem (see Rader [13]) which states that there is an upper semicontinuous utility function $u$ for every upper semicontinuous total preorder $\precsim$ on a second countable topological space $(X, \tau)$.

The following characterization of an i.o. completely useful topology in the metrizable case is similar to Corollary 4.5 in Bosi and Herden [1] and to the Theorem in Estévez and Hervés [7].

**Proposition 3.4** *Let $\tau$ be a metrizable topology on a set $X$. Then the following conditions are equivalent:*

(i) *$\tau$ is i.o. completely useful;*

(ii) *$\tau$ is separable.*

*Proof.* (i) ⇒ (ii). If $\tau$ is an i.o. completely useful and metrizable topology on a set $X$ then from Lemma 3.1 we have that $\tau$ is completely useful and therefore it is separable from Bosi and Herden [1, Corollary 4.5].
(ii) ⇒ (i). If $\tau$ is a metrizable and separable topology on a set $X$ then $\tau$ is second countable and therefore it is i.o. completely useful by Lemma 3.2. This consideration completes the proof. □

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