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Holomorphic Triples on Singular Genus One Curves

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Abstract. Here we study the α-stability of coherent systems and holomorphic triples on a singular genus 1 curve.

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Let C be an integral complex projective curve such that $p_a(C) = 1$. Hence either C is a smooth elliptic curve or $\mathbf{P}^1$ is the normalization of C and C has a unique singular point which is either an ordinary node or an ordinary cusp. Since every result proved in this paper is known when C is smooth, we will always assume that C is singular and call Q its singular point. Here we will just collect the results on coherent systems and holomorphic triples which may be obtained with easy modifications of the proofs of the corresponding results proved in [8] and [9]. For background and definitions of coherent systems see [5], [6] and [8] and references therein. We just recall in our set-up the definition of holomorphic triples ([4], [7], [9]). A holomorphic triple $T = (E_1, E_2, \phi)$ on C is given by torsion free sheaves E_i on C together with a morphism $\phi : E_2 \to E_1$. For all $\alpha \in \mathbb{R}$ set $\mu_\alpha(T) = (\deg(E_1) + \deg(E_2) + \alpha \cdot \mathrm{rank}(E_2))/(\mathrm{rank}(E_1) + \mathrm{rank}(E_2))$. A subtriple (F_1, F_2, ψ) is given by subsheaves $F_i \subseteq E_i$, $i = 1, 2$, such that $\phi(F_2) \subseteq F_1$, while $\psi = \phi|F_2$; we allow the case $F_2 = \{0\}$, $F_1 \neq \{0\}$ and in this case we set $\mathrm{rank}(F_2) = 0$ and $\deg(F_2) = 0$. T is α-stable (resp. α-semistable) if $\mu_\alpha(T') < \mu_\alpha(T)$ (resp. $\mu_\alpha(T') \le \mu_\alpha(T)$ for all proper subtriples T' of T.We will heavily use the more elementary parts of the unpublished Ph. D. thesis [10]. As in the smooth case for all coprime integers m, d and all $L \in \mathrm{Pic}^d(C)$ there is a unique rank m stable vector bundle E on C such that

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$\det(E) \cong L$ ([10], Th. 5.1). Hence for all integers r, d such that $r \geq 1$ there is a rank r polystable vector bundle E on C such that the indecomposable factors of E are pairwise non-isomorphic.

Notation 1. For every sheaf F on C and any linear subspace $V \subseteq H^0(C, F)$ let $e_{F,V} : V \otimes \mathcal{O}_C \to F$ denote the evaluation map. Set $e_F := e_{F,H^0(C,F)}$.

Theorem 1. *Fix integers n, d, k such that $d > n \geq k > 0$. Let E be a rank n polystable vector bundle on C such that $\deg(E) = d$ and its indecomposable factors are pairwise non-isomorphic. Hence $h^0(C, E) = d$. Fix a general linear subspace $V \subseteq H^0(C, E)$.*

(a) *If $k < n$, then $e_{F,V}$ is injective, Coker$(e_{F,V})$ is a polystable vector bundle with pairwise non-isomorphic indecomposable factors and the coherent system (E, V) is α-stable for all $0 < \alpha < d/(n-k)$.*

(b) *If $k = n$, then $e_{E,V}$ is injective and the coherent system (E, V) is α-stable for all $\alpha > 0$.*

Theorem 2. *Fix integers $n_1 > n_2 > 0$, $d_1 > 0$, and a polystable vector bundle E_1 on C such that its indecomposable factor are pairwise non-isomorphic, rank$(E_1) = n_1$ and $\deg(E_1) = d_1$. Fix a general $\phi \in H^0(C, Hom(\mathcal{O}_C^{n_2}, E_1))$. Then the triple $(E_1, \mathcal{O}_C^{n_2}, \phi)$ is α-stable for all α such that $d_1/n_1 < \alpha < 2d_1/(n_1 - n_2)$.*

Remark 1. Fix a coherent system (E, V) on C and real numbers $\alpha > \beta$. It is easy to check that if (E, V) is α-stable and β-stable, then it is γ-stable for all γ such that $\alpha > \gamma > \beta$. The same statement is easily seen to be true for a triple on C.

Remark 2. Fix positive integers d, n and a semistable vector bundle E on C such that rank$(E) = n$ and $\deg(E) = d$. Since E is semistable and $\deg(E) > 0$, we have $h^1(C, E) = 0$. Hence $h^0(C, E) = d$ (Riemann-Roch). Now assume $d \geq n$ and that F is polystable and its indecomposable factors are pairwise non-isomorphic. Fix $P \in C_{reg}$. Hence $E(-P)$ is a semistable vector bundle with rank n and degree $d - n$. Hence $h^1(C, E(-P)) = 0$ if either $d > n$ or $d = n$ and P is general. Hence e_E is surjective at each point of C_{reg} if $d > n$, while e_E is injective if $d = n$. Thus for any $d \geq n$ and any positive integer $m \leq n$ the map $e_{E,V}$ is injective for a general m-dimensional linear subspace V of $H^0(C, E)$. Now assume $d > n$. We want to check that E is spanned. We just saw that it is sufficient to prove that E is spanned at Q. The proof in [10], Th. 1.3, gives the existence of a polystable vector bundle F such that rank$(F) = n$, $\deg(F) = d$, with pairwise indecomposable factors and such that the n-dimensional vector space $H^0(C, F)$ spans F at Q. We may find F such that there is a map $j : F \to E$ whose restriction to the fiber over Q has rank n (see the proof of [1], part (b) of Theorem 1). Since e_F is surjective at Q, e_E is surjective at Q.

Lemma 1. *Fix integers $n > k > 0$ and $d \geq k$. Let E be rank n polystable vector bundle on C such that* $\deg(E) = d$ *and its indecomposable factors are pairwise non-isomorphic. Then $h^0(C, E) = d$. Let $V \subseteq H^0(C, E)$ be a general k-dimensional linear subspace. Then $e_{F,V}$ is injective and Coker($e_{F,V}$) is a polystable vector bundle with pairwise non-isomorphic indecomposable factors.*

Proof. The first part of Remark 2 gives $h^0(C, E) = d$. Remark 2 implies the injectivity of $e_{E,V}$ and that Coker($e_{F,V}$) is locally free. Since Coker($e_{F,V}$) is a quotient of E, $\mu_-(\mathrm{Coker}(e_{F,V})) \geq \mu(E) > 0$. Thus $h^0(C, Hom(\mathrm{Coker}(e_{F,V}), V \otimes \mathcal{O}_C)) = 0$. This equality and [3], Remark 2.6, implies the lemma ([1]). □

Lemma 2. *Fix integers $n > k > 0$ and $d \geq k$. Let E be rank n polystable vector bundle on C such that* $\deg(E) = d$ *and its indecomposable factors are pairwise non-isomorphic. Then $h^0(C, E) = d$. Let $V \subseteq H^0(C, E)$ be a general k-dimensional linear subspace. Then $e_{F,V}$ is surjective and Ker($e_{F,V}$) is a polystable vector bundle with pairwise non-isomorphic indecomposable factors.*

Proof. The first part of Remark 2 gives $h^0(C, E) = d$. The proof of the second part is the dual of the proof of Lemma 2. □

Proof of Theorem 1 when $k < n$. The first part of Remark 2 gives $h^0(C, E) = d$. Fix (E, V). We will first show that (E, V) is α-stable when $0 < \alpha \ll 1$. Assume that this is not true and take a coherent subsystem (F_α, W_α) which α-destabiles (E, V). Since E is polystable, every destabilizing subsheaf of E is a direct factor of E and in particular it is locally free. Since the indecomposable factors of E are pairwise non-isomorphic, E has only finitely many destabilizing subsheaves, each of them ahas slope d/n and each of them is a direct factor of E. Properness for semistable coherent systems implies that the sheaves F_α for smaal α are deformations of a direct factor of E. Hence F_α is locally free for $0 < \alpha \ll 1$. exactly as in the smooth case ([8], Th. 4.1) for $0 < \alpha \ll 1$ each F_α must be a direct factor of E. as in the proof of [8], Th. 4.1, we get a contradiction. By remark 1 it is sufficient to prove the α-stability of (F, V) when $d/(n-k) - \epsilon \leq \alpha < d/(n-k)$ and $0 < \epsilon \ll 1$. The proof of [8], Th. 5.2 (whuch in turn is a careful modification of □

Proof of Theorem 1 when $k = n$. The first part of Remark 2 gives $h^0(C, E) = d$. Since $d > n$, $V \neq H^0(C, E)$. Since E has finitely many direct factors, the generality of V implies $\dim(V \cap H^0(C, G)) < \mathrm{rank}(G)$ for every proper direct factor G of E. The evaluation map $e_{E,V}$ is injective. The last two sentences and the polystability of E imply the α-stability of (E, V) for all $\alpha > 0$. □

Proof of Theorem 2. Use the proof of [9], Th. 4.3, with the observation that ϕ is injective and that its cokernel is locally free, polystable and with pairwise non-isomorphic indecomposable factors (Lemma 1). □

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