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Projective Subschemes with Many Linear Automorphisms

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Abstract. Here we study projective subschemes of $\mathbf{P}^n$ with many automorphisms, pointing out some peculiarities of the positive characteristic case.

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1. Introduction

We work over an algebraically closed field $\mathbb{K}$. We first introduce 3 of our players. Fix $P \in \mathbf{P}^n$. Set $G := \mathrm{Aut}(\mathbf{P}^n)$ and $G(P) := \{f \in G : f(P) = P\}$. For any integer $m > 0$ let $(m, n)P$ denote the $(m-1)$-th infinitesimal neighborhhod of P in $\mathbf{P}^n$, i.e. the closed subscheme of $\mathbf{P}^n$ with ${\mathcal{I}_P}^m$ as its ideal sheaf. Hence $((m, n)P)_{red} = \{P\}$ and $\mathrm{length}((m, n)P) = \binom{n+m-1}{n}$. $(m, n)P$ is called a fat point of $\mathbf{P}^n$ with multiplicity m. Now assume $p := \mathrm{char}(\mathbb{K}) > 0$, and fix a p-power q and any $P \in \mathbf{P}^n$. Let $[q, n]P$ denote the closed subscheme of $\mathbf{P}^n$ whose ideal is generated by all L^q, where L is a homogeneous degree 1 form vanishing at P. Hence $([q, n]P)_{red} = \{P\}$. Since $(L + M)^q = L^q + M^q$ for all linear forms L, M, $[q, n]P$ is defined by the vanishing of the q-powers of any n linearly independent linear forms vanishing at P. Fix homogeneous coordinates $x_0, \dots, x_n$ such that $P = (1; 0, \dots; 0)$. We get that $[q, n]P$ is defined by the monomial equations ${x_i}^q = 0$, $1 \le i \le q$. Thus $\mathrm{length}([q, n]P) = q^n$. We will say that $[q, n]P$ is a p-fat point of $\mathbf{P}^n$ with multiplicity q. Obviously, both mP and $[q, n]P$ are $G(P)$-invariants. In section 2 we will prove the following result.

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Theorem 1. *Fix* $P \in \mathbf{P}^n$ *and a zerodimensional scheme* $Z \subset \mathbf{P}^n$ *such that* $Z_{red} = \{P\}$ *and* $f(Z) = Z$ *for all* $f \in G(P)$*. Let* $m > 0$ *be the maximal integer such that* $(m,n)P \subseteq Z$*.*

(a) *If* $char(\mathbb{K}) = 0$*, then* $Z = (m,n)P$*.*

(b) *Assme* $p := char(\mathbb{K}) > 0$*. If* $m < p$*, then* $Z = (m,n)P$*. If* $m = p$*, then* $(m,n)P \subseteq Z \subseteq [p,n]P$*.*

Now we consider closed subschemes $Z \subset \mathbf{P}^n$ such that $\dim(Z_{red}) > 0$. Fix any closed subscheme $Z \subset \mathbf{P}^n$ and any $P \in \mathbf{P}^n$. Set $G_Z := \{f \in G : f(Z) = Z\}$, $G(P,Z) := G_Z \cap G(P)$ and $G(Z) := \cap_{Q \in Z_{red}} G(Q,Z)$. Let $M \subsetneqq \mathbf{P}^n$ a non-empty linear subspace. Notice that $f(P) = P$ for all $f \in G(M)$ and all $P \in M$. Hence every closed subscheme of M is $G(M)$-invariant. The next result considers the closed subschemes Z of $\mathbf{P}^n$ which are $G(M)$-invariant. Notice that if Z is a $G(M)$-invariant closed subscheme of $\mathbf{P}^n$, then Z_{red} is $G(M)$-invariant and hence either $Z = \mathbf{P}^n$ or $Z_{red} \subseteq M$.

Theorem 2. *Let* $M \subsetneqq \mathbf{P}^n$ *a non-empty proper linear subspace and* $Z \subsetneqq \mathbf{P}^n$ *a* $G(M)$*-invariant closed subscheme. Let* $W \subseteq Z$ *be a closed primary component of* Z *(even an embedded one) for any primary decomposition of* Z *(or at least for a local one). Set* $T := Z_{red}$*. Hence* T *is irreducible and* $T \subseteq M$*. Let* P *be a general point of* T*. Let* m *be the only positive integer such that for a general line* $D \subset \mathbf{P}^n$ *such that* $P \in D$ *the connected component of the scheme* $D \cap W$ *supported by* P *has length* m*. Assume either* $char(\mathbb{K}) = 0$ *or* $char(\mathbb{K}) > m$*. Let* $T^{(m-1)}$ *denote the* $(m-1)$*-th infinitesimal neighborhood of* T *in* $\mathbf{P}^n$*, i.e. the closed subscheme of* $\mathbf{P}^n$ *with* ${\mathcal{I}_T}^m$ *as its ideal sheaf. Then* $T^{(m-1)} \subseteq W$ *and* $\mathcal{I}_{T^{(m-1)},P}$ *may be used instead of* $\mathcal{I}_{W,P}$ *in a primary decomposition of the ideal* $\mathcal{I}_{Z,P}$ *of the locally ring* $\mathcal{O}_{\mathbf{P}^n,P}$*.*

Take the set-up of Theorem 2. Notice the connected component of the scheme $D \cap W$ supported by P has length m. If W is an embedded component of Z, then $T^{(m-1)} \cap U \neq Z \cap U$ for every open neighborood U of P in $\mathbf{P}^n$. because Z_{red} has an irreducible component strictly containing T. If T is a connected component of Z_{red}, then there is an open neighborhood U of P in $\mathbf{P}^n$ such that $Z \cap U = W \cap U = T^{(m-1)} \cap U$.

2. THE PROOFS

Proof of Theorem 1. Fix homogeneous coordinates $x_0, \dots, x_n$ such that $P = (1;0,\dots;0)$. Write $z_i := x_i/x_0$ for $1 \le i \le n$. For any homogeneous form $f(x_0,\dots,x_n)$, $f \neq 0$, vanishing at P let $\widetilde{f}(z_1,\dots,z_n) := f(1,z_1,\dots,z_n)$ the associated non-homogeneous polynomial. Take f as above for which the Taylor series at $(0,\dots,0)$ of $\widetilde{f}$ has non-zero minimal degree term $\bar{f}$ with minimal degree. For every $\lambda \in \mathbb{K}\backslash\{0\}$ let $h_\lambda : \mathbf{P}^n \to \mathbf{P}^n$ by defined by the formula $h_\lambda(x_0;x_1;\dots;x_n) = (\lambda x_0;x_1;\cdots;x_n)$. Since $G(Z) = G(P)$, we get that $f(x_0,\lambda x_1,\dots,\lambda x_n)$ vanishes on Z. Hence $\widetilde{f}(\lambda z_1,\dots,\lambda z_n)|Z \equiv 0$ for every $\lambda \in \mathbb{K}\backslash\{0\}$. Hence $\bar{f}|Z \equiv 0$, i.e. in this affine chart Z is defined by homogeneous equations. Let u be the dominant monomial appearing in $\bar{f}$ with non-zero coefficient with respect to the lexicographic order of the variables $z_1, \dots, z_n$. Taking the linear transformation $(x_0; x_1; \dots; x_n) \mapsto (x_0, \mu_1 x_1; \dots; \mu_n x_n)$ for suitable $\mu_1, \dots, \mu_n$, we get that every monomial appearing in $\bar{f}$ with non-zero coefficient vanishes on Z. Thus Z is defined by monomial equations. Since $f(Z) = Z$ for every $f \in G(P)$, this is true for any choice of homogeneous coordinates such that $P = (1; 0; \dots; 0)$. Let c be the degree of a monomial in $z_1, \dots, z_n$ vanishing on Z. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > c$. First assume that z_i^c is such a monomial. Since Z in $G(P)$-invariant, we get $L^c|Z \equiv 0$ for every homogeneous linear form in the variables $z_1, \dots, z_n$. Since either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > c$, every degree monomial in $z_1, \dots, z_n$ vanishes on Z, i.e. $Z = (c, n)P$. Now assume that a degree c monomial vanishing in Z involves at least 2 variables, say z_i and z_j with $i < j$. Use the map $z_i \mapsto z_i$ and $z_j \mapsto z_i + \lambda z_i$ and that Z is defined by monomials for any system of coordinates. We get that there is a degree c monomial vanishing on Z and in which z_j does not appear. And so on. If $\mathrm{char}(\mathbb{K}) = c$ we only get $Z \subseteq [p, n]P$. $\square$

Proof of Theorem 2. Take a linear subspace $V \subset \mathbf{P}^n$ such that $V \cap M = \{P\}$ and $\langle V \cup M \rangle = \mathbf{P}^n$. Set $G\{V\} := \mathrm{Aut}(V)$ and $G\{V, P\} := \{f \in G_\{V\} : f(P) = P\}$. The scheme $Z \cap V$ is $G\{V, P\}$-invariant. Theorem 1 implies $Z \cap V = (m, \dim(M))P$. Hence $T^{(m-1)} \subseteq W$. Both $T^{(m-1)}$ is primary, because in a commutative local ring the m-th power of the maximal ideal is primary ([1], Prop. 4.2). Notice that $\deg(T^{(m-1)}) = \binom{\dim(V)+m-1}{\dim(V)} = \deg(W)$. Hence $T^{(m-1)}$ and W are (in a neighborhood of a general $P \in T$) two allowable primary components of $\mathcal{I}_{Z,P}$. $\square$

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