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Uniform Vector Bundles on $\mathbf{P}^N$ as Syzygy Bundles

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Abstract. Fix integers N, d, v such that $N \geq 2$ and $d > 0$. For any v-dimensional linear subspace $V \subseteq H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(d))$ spanning $\mathcal{O}_{\mathbf{P}^N}(d)$ let $\ell_{N,d;V} : V \otimes \mathcal{O}_{\mathbf{P}^N} \to \mathcal{O}_{\mathbf{P}^N}(d)$ be the evaluation map and $W_{N,d;V} := \mathrm{Ker}(\ell_{N,d;V})$ the associated syzygy bundle. Here we show that if $\dim(V) \geq d + 2N$ and V is general, then $W_{N,d;V}$ is uniform, i.e. its restriction to all lines has the same splitting type.

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Here we study syzygy bundles on $\mathbf{P}^N$ and found examples of them which are uniform vector bundle. For their connection with Artinian algebras, see [1], [2], [6], [9] and Remark 7.

Let A be a rank r vector bundle on $\mathbf{P}^1$. Let $a_1 \geq \cdots \geq a_r$ be the splitting type of A. A is said to be *rigid* if $a_r \geq a_1 - 1$.

Remark 1. Let A be a rank r vector bundle on $\mathbf{P}^1$. It is easy to check that A is rigid if and only if for every integer t either $h^0(\mathbf{P}^1, A(t)) = 0$ or $h^1(\mathbf{P}^1, A(t)) = 0$.

Let A be a vector bundle on $\mathbf{P}^N$, $N \geq 2$. A is said to be uniform if for all lines $L, D \subset \mathbf{P}^N$ the vector bundles $A|L$ and $A|D$ have the same splitting type. Obviously, every homogeneous vector bundle is uniform. If $\mathrm{rank}(A) \leq N$ and A is uniform, then A is either a direct sum of line bundles or a twist of the cotangent bundle or a twist of the tangent bundle and hence it is homogeneous

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(see [4] and references their for much more) If $N = 2$ and $r = 3$, a similar classification is known ([4]), but for all ranks $r \geq 4$ a general rank r stable vector bundle on $\mathbf{P}^2$ with non-integral slope is uniform ([3]). Let A be a vector bundle on $\mathbf{P}^N$. The generic splitting type of A is the splitting type of the vector bundle $A|L$, where L is a general line of $\mathbf{P}^N$. A line $D \subset \mathbf{P}^N$ is said to be a jumping line of A if the vector bundle $A|D$ has a splitting type different from the generic splitting type of A.

Notation 1. Fix positive integers integers N, d. Let $\ell_{N,d} : H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(d)) \otimes \mathcal{O}_{\mathbf{P}^N} \to \mathcal{O}_{\mathbf{P}^N}(d)$ be the evaluation map. Set $W_{N,d} := \mathrm{Ker}(\ell_{N,d})$. Hence $W_{N,d}$ is a homogeneous vector bundle, $\mathrm{rank}(W_{N,d}) = \binom{N+d}{N} - 1$ and $c_1(W_{N,d}) = d$. For any linear subspace $V \subseteq H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(d))$, let $\ell N, d, V : V \otimes \mathcal{O}_{\mathbf{P}^N} \to \mathcal{O}_{\mathbf{P}^N}(d)$ be the evaluation map. If $\ell_{N,d,V}$ is surjective, i.e. if V spans $\mathcal{O}_{\mathbf{P}^N}$, set $W_{N,d,V} := \mathrm{Ker}(\ell_{N,d})$. Hence $W_{N,d,V}$ is a homogeneous vector bundle, $\mathrm{rank}(W_{N,d,V}) = \dim(V) - 1$ and $c_1(W_{N,d,V}) = d$. For any line $L \subset \mathbf{P}^N$ and any linear subspace $V \subseteq H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(d))$ set $O(N, d, -L) := H^0(\mathbf{P}^N, \mathcal{I}_L(d))$ and $V(-L) := V\cap)(N, d, -L)$. $O(N, d, -L)$ is a codimension $(d + 1)$ linear subspace of $H^0(\mathbf{P}^N, \mathcal{I}_L(d))$. For every integer v such that $0 \leq v \leq \binom{N+d}{N}$ let $G(v; N, d)$ denote the Grassmannian of all v-dimensional linear subspaces of $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(d))$. Let $G(v; N, d)$ be the open subset of $G(v; N, d)$ formed by the v-dimensional linear subspaces spanning $\mathcal{I}_L(d)$. $G(v; N, d)' \neq \emptyset$ if and only if $N + 1 \leq v \leq \binom{N+d}{N}$. For any integer $x \geq 0$ set $B(v; N, d; L, x) := \{V \in G(v; N, d, x) := \dim(V(-L)) = x\}$ and
$B(v; N, d; L) = \cup_{x>\min\{0,v-d-1\}} B(v; N, d; L, x)$.

Here we prove the following results.

Theorem 1. *Fix integers d, N, v such that $d > 0$, $N \geq 2$ and $d + 1 \leq v \leq \binom{N+d}{N}$. Let V be a general v-dimensional linear subspace of $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(d))$. Then $W_{N,d,V}|L$ is rigid for a general line $L \subset \mathbf{P}^N$.*

Theorem 2. *Fix integers N, d, v such that $N \geq 2$ and $d + 2N \leq v \leq \binom{N+d}{N}$. Let V be a general element of $G(v; N, d)$. Then $W_{N,d,V}$ is uniform and $W_{N,d,V} \cong \mathcal{O}_L(-1)^{\oplus d} \oplus \mathcal{O}_L^{\oplus(v-d-1)}$ for every line $L \subset \mathbf{P}^N$*

Remark 2. For any line $D \subset \mathbf{P}^N$ the vector bundle $W_{N,d}|D$ is isomorphic to a direct sum of d line bundles of degree -1 and $\binom{N+d}{N} - 1 - d$ line bundles of degree 0.

Remark 3. Let $V \subseteq H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$ any linear subspace spanning $\mathcal{O}_{\mathbf{P}^1}(d)$. Since $h^0(\mathbf{P}^1, W_{1,d,V}) = 0$, every rank 1 direct factor of $W_{1,d,V}$ has degree < 0.

Remark 4. Fix integers d, N, v such that $d > 0$, $N \geq 2$ and $0 \leq v \leq \binom{N+d}{N}$, and a line $L \subset \mathbf{P}^N$. Let V be a general v-dimensional linear subspace of $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(d))$. Then $\dim(V(-L)) = \max\{0, v - d - 1\}$. Hence $\dim(V/V(-L)) = \min\{d + 1, v\}$.

Remark 5. Fix integers $N \geq 2$ and $d > 0$, a line $L \subset \mathbf{P}^N$ and a linear subspace $V \subseteq H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(d))$. Set $w := \dim(V/V(-L))$ and $v := \dim(V)$. By Remark 3 the splitting type of the vector bundle $W_{N,d,V}$ contains $w-1$ strictly negative integers and 0 with multiplicity $v-w$. Notice that if $v = w = d+1$, the value of $c_1(W_{N,n,d}$ implies that $W_{N,d,V}|L \cong \mathcal{O}_L^{\oplus d}$.

Proof of Theorem 1. The case $v = d+1$ is true by Remark 4 and the last sentence of Remark 5. This is the only case in which $W_{N,d,V}|L$ is semistable. The general case follows from Remarks 4 and 5 and the fact that $c_1(W_{N,d,V}) = -d$. □

Remark 6. Fix integers d, v such that $2 \leq v \leq d+1$. For all sequence of integers $a_1, \dots, a_{v-1}$ such that $-1 \geq a_1 \geq \cdots \geq a_{v-1}$ and $a_1 + \cdots + a_{v-1} = -d$ let $G(v; 1, d; a_1, \dots, a_{v-1})$ denote the set of all $V \in G(v; 1, d)$ such that V spans $\mathcal{O}_{\mathbf{P}^1}(d)$ and $W_{1,d,V}$ has splliting type $a_1, \dots, a_{v-1}$. $G(v; 1, d; a_1, \dots, a_{v-1}) \neq \emptyset$ for all $a_1, \dots, a_{v-1}$ ([10]). For a general $V \in G(v; 1, d)$ the vector bundle $W_{1,d,V}$ is rigid ([10]).

Remark 7. Assume $N = 2$ and take any syzygy bundle $F := A_{2,d,V}$. Let $A = \oplus A_i$ be the Artinian graded algebra associated to F ([2], Prop. 2.1). Fix any line $L := \{\ell = 0\} \subset \mathbf{P}^2$ and assume that $F|L$ is rigid. The proof of [2], part (1) of Theorem 2.2, shows that A has the Weak Lefschetz Property and for each i the multiplication map $\times \ell_i : A_i \to A_{i+1}$ by ℓ has maximal rank. More precisely, if F has splitting type $a_1 \geq \cdots \geq a_r$ with either $a_1 = a_r$ or $a_1 = a_r - 1$, then $\times \ell_i : A_i \to A_{i+1}$ is injective if $i \leq -a_1$ and it is surjective if $i \geq -a_r$. Now we drop the assumption that $F|L$ is rigid and call again $a_1 \geq \cdots \geq a_r$ its splitting type. The same proof shows that $\times \ell_i : A_i \to A_{i+1}$ is injective if $i \leq -a_1$ and it is surjective if $i \geq -a_r$.

Remark 8. Fix integers N, d, v, x such that $N \geq 2$, $d > 0$ and $N+1 \leq v \leq binom N + dN$ and $\min\{0, v-d-1\} < x \leq v$. Fix a line $L \subset \mathbf{P}^N$. The algebraic sets $B(v; N, d; L, x)$ and $B(v; N, d; L, x)'$ are open subsets of a suitable Schubert cycle of $G(v; N, d)$, Hence they are either irreducible or empty. We have $\dim(B(v; N, d; L, x) = x(\binom{N+d}{N} - 1 - d - x) + (\binom{N+d}{N} - v)(v - x)$, i.e $B(v; N, d; L, x)$ has codimension $x(x+d+1-v)$ in $G(v; N, d)$ ([5], p. 196, or [7], p. 149).

Proof of Theorem 2. Fix an integer $x > v-d-1$ and a line $L \subset \mathbf{P}^N$. By Remark 8 $B(v; N, d; L, x)$ has codimension $x(x+d+1-v) > 2N-2$ in $G(v; N, d)$. Hence $B(v; N, d; L)$ has codimension ate least $2N-1$ in $G(v; N, d)$. Notice that $W_{N,d,V} \cong \mathcal{O}_L(-1)^{\oplus d} \oplus \mathcal{O}_L^{\oplus (v-d-1)}$ if $V \in G(v; N, d) \backslash B(v; N, d; L)$. Since the Grassmannian of all lines in $\mathbf{P}^N$ has dimension $2N-2$, we get the result. □

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