Riemann-Stieltjes Operators between Weighted Bloch and Weighted Bergman Spaces

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Abstract
In this paper, Riemann-Stieltjes operators between weighted Bloch and weighted Bergman spaces are considered. We characterize boundedness and compactness of these operators using certain growth properties of holomorphic symbols.

Keywords: weighted Bergman spaces, weighted Bloch spaces, Riemann-Stieltjes operator, Carleson measure

1 Introduction
Let \( \mathbb{D} \) be the open unit disk in the complex plane \( \mathbb{C} \). Let \( g : \mathbb{D} \rightarrow \mathbb{C} \) be a holomorphic map. Denote by \( H(\mathbb{D}) \) the space of holomorphic functions on \( \mathbb{D} \). For \( f \in H(\mathbb{D}) \), the Riemann-Stieltjes operator induced by \( g \) is defined by
\[
T_g f(z) = \int_0^z f(\zeta)dg(\zeta) = \int_0^1 f(tz)zg'(tz)dt, \quad z \in \mathbb{D}.
\]
The Riemann-Stieltjes operator can be viewed as a generalization of Cesaro operator defined by
\[
Tf(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1-w}d(w), \quad z \in \mathbb{D}.
\]

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Ch. Pommerenke [7] initiated the study of Riemann-Stieltjes operator on $H^2$, where he showed that $T_g$ is bounded on $H^2$ if and only if $g$ is in $BMOA$. This was extended to other Hardy spaces $H^p$, $1 \leq p < \infty$, in [1] and [2] where compactness of $T_g$ on $H^p$ and Schatten class membership of $T_g$ on $H^2$, was also completely characterized in terms of the symbol $g$. Similar questions on weighted Bergman spaces were considered by A. Aleman and A. G. Siskakis in [3].

Recently, several authors have studied Riemann-Stieltjes operators on different spaces of analytic functions. For example, one can refer to ([5] [8] [9] [10] [11] and [12]) for the study of these operators on Bergman spaces, Dirichlet spaces, BMOA and VMOA and related references therein. In this paper we characterize boundedness and compactness of Riemann-Stieltjes operators between weighted Bloch and weighted Bergman spaces.

2 Preliminaries

In this section we review the basic concepts of weighted Bergman spaces $A^p_\alpha$ and weighted Bloch spaces $B^\alpha$ and collect some essential facts that will be needed throughout the paper.

2.1. Weighted Bergman Spaces. Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that area of $\mathbb{D}$ is 1. For each $\alpha \in (-1, \infty)$, we set $d\nu_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$, $z \in \mathbb{D}$. Then $d\nu_\alpha$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$ the weighted Bergman space $A^p_\alpha$ is defined as

$$A^p_\alpha = \{ f \in H(\mathbb{D}) : ||f||_{A^p_\alpha} = \left( \int_{\mathbb{D}} |f(z)|^p d\nu_\alpha(z) \right)^{1/p} < \infty \}.$$

Note that $||f||_{A^p_\alpha}$ is a true norm only if $1 \leq p < \infty$ and in this case $A^p_\alpha$ is a Banach space. For $0 < p < 1$, $A^p_\alpha$ is a non-locally convex topological vector space and $d(f, g) = ||f - g||_{A^p_\alpha}$ is a complete metric for it. The growth of functions in the weighted Bergman spaces is essential in our study. To this end, the following sharp estimate will be useful. (see [7] p. 53.). It tells us how fast an arbitrary function from $A^p_\alpha$ grows near the boundary.

Let $f \in A^p_\alpha$. Then for every $z$ in $\mathbb{D}$, we have

$$|f(z)| \leq \frac{||f||_{A^p_\alpha}}{(1 - |z|^2)^{(2+\alpha)/p}}$$

with equality holds if and only if $f$ is a constant multiple of the function

$$k_\alpha(z) = \left( \frac{1 - |z|^2}{1 - \overline{a}z} \right)^{(2+\alpha)/p}.$$

It can be easily shown that $||k_\alpha||_{A^p_\alpha} \approx 1$.

2.1. Weighted Bloch Spaces. For $\alpha > 0$, let $B^\alpha$ consists of all
analytic functions $f$ on $\mathbb{D}$ satisfying the condition
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.
\]
Note that $B^1 = B$, the usual Bloch space. For $f \in B^\alpha$ define
\[
||f||_{B^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.
\]
With this norm $B^\alpha$ is a Banach space. Integrating the estimate
\[
|f'(z)| \leq ||f||_{B^\alpha} (1 - |z|^2)^\alpha,
\]
we obtain
\[
|f(z) - f(0)| \leq \int_0^1 |z||f'(tz)|dt \leq ||f||_{B^\alpha} \int_0^1 \frac{|z|}{(1 - t|z|^2)^\alpha}dt,
\]
for all $z \in \mathbb{D}$. In case $0 < \alpha < 1$, the integral on the right is uniformly bounded by the constant $\int_0^1 (1 - t)^{-\alpha}dt$, and it follows that $B^\alpha \subset H^\infty$. It is easy to check that in this case the linear space $B^\alpha$ is an algebra. In fact, Hardy and Littlewood, have shown that for $0 < \alpha < 1$, the space $B^\alpha$ consist of all functions $f$ analytic on $\mathbb{D}$ satisfying the Lipschitz condition
\[
|f(z) - f(w)| \leq |z - w|^{1-\alpha},
\]
for all $z, w \in \mathbb{D}$ (see [4]).

In case $1 < \alpha < \infty$, the above estimate implies
\[
|f(z) - f(0)| \leq \frac{||f||_{B^\alpha}}{\alpha - 1} \frac{1}{(1 - |z|^2)^{\alpha-1}}, \tag{2.3}
\]
while for $\alpha = 1$ it is well known that the following hold ([6] and [14]).
\[
|f(z) - f(w)| \leq ||f||_B \beta(z, w) \tag{2.4}
\]
for $f \in B$, where
\[
\beta(z, w) = \frac{1}{2} \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}
\]
is the Bergman metric on $\mathbb{D}$. From (2.4), it follows that for $f \in B$,
\[
|f(z)| \leq \frac{1}{\log 2} ||f||_B \log \left( \frac{2}{1 - |z|^2} \right). \tag{2.5}
\]
Throughout this paper we fix some positive radius $0 < r < \infty$ and consider disks $D(z, r)$ in the Bergman metric. The set
\[
D(z, r) = \{ w \in \mathbb{D} : \beta(z, w) < r \}, \quad z \in \mathbb{D},
\]
is called hyperbolic disk or Bergman disk of radius $r$ about $z$. It is well known that $D(z, r)$ is a Euclidean disk whose Euclidean center and Euclidean radius are
\[
\frac{(1 - s^2)z}{(1 - s^2|z|^2)} \quad \text{and} \quad \frac{(1 - |z|^2)s}{(1 - s^2|z|^2)}.
\]
where $s = \tanh r \in (0, 1)$, respectively. For fixed $r > 0$, the area of $D(z, r)$ in $\mathbb{D}$ has the estimation;
\[
|D(z, r)|_A = \int_{D(z, r)} dA(w) \approx (1 - |z|^2)^2.
\]
For fixed $r > 0$, it is known that if $w \in D(z, r)$, then
\[
|1 - z\bar{w}| \approx (1 - |z|^2) \quad \text{and} \quad |D(w, s)|_A \approx C|D(z, r)|_A.
\]
Following lemma lists additional properties of the hyperbolic disks.

**Lemma 2.2.** [7] Fix $r, \ 0 < r < \infty$. There exists a positive integer $M$ and a sequence $\{a_n\}$ in $\mathbb{D}$ such that :

(i) The disk $\mathbb{D}$ is covered by $\{D(a_n, r)\}_n$.

(ii) Every point in $\mathbb{D}$ belongs to at most $M$ sets in $\{D(a_n, 2r)\}_n$.

(iii) If $n \neq m$, then $\beta(a_n, a_m) \geq \frac{r}{7}$.

We shall use these estimates in the proofs of the Theorems below. For general background of weighted Bergman spaces $A^p_\alpha$ and weighted Bloch spaces, one may consult [13] and [14] and the references therein.

**3 Riemann-Stieltjes operators from weighted Bergman spaces $A^p_\alpha$ into weighted Bloch spaces $B^\alpha$**

In this section we characterize boundedness and compactness of Riemann-Stieltjes operators from weighted Bergman spaces $A^p_\alpha$ into weighted Bloch spaces $B^\alpha$.

The following Theorem characterizes Riemann-Stieltjes operators from weighted Bergman spaces $A^p_\beta$ into weighted Bloch spaces $B^\alpha$.

**Theorem 3.1.** Let $1 \leq p < \infty, -1 < \beta < \infty, \alpha > 0$ and $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic map. Then the Riemann-Stieltjes operator $T_g$ maps $A^p_\beta$ boundedly into $B^\alpha$ if and only if
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{(p\alpha - \beta - 2)/p} |g'(z)| < \infty.
\]
Proof. First suppose that
\[ M = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{(p\alpha - \beta - 2)/p} |g'(z)| < \infty. \]

By (2.1), we have
\[ |f(z)| \leq \frac{||f||_{A_p^\beta}}{(1 - |z|^2)^{(\beta + 2)/p}} \]
for all \( z \in \mathbb{D} \). Thus
\[
||T_g f||_{B^\alpha} = |T_g f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |(T_g f)'(z)| \\
= \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |g'(z)f(z)| \\
\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{(p\alpha - \beta - 2)/p} |g'(z)| ||f||_{A_p^\beta} \\
= M ||f||_{A_p^\beta},
\]

hence \( T_g \) maps \( A_p^\beta \) boundedly into \( B^\alpha \).

Conversely, suppose \( T_g \) maps \( A_p^\beta \) boundedly into \( B^\alpha \). Fix a point \( a \) in \( \mathbb{D} \) and consider the function
\[
f_a(z) = \left( \frac{1 - |a|^2}{(1 - \overline{a}z)^2} \right)^{(\beta + 2)/p}.
\]
Then \( f_a \) is a function of unit norm in \( A_p^\beta \). Since \( T_g \) maps \( A_p^\beta \) boundedly into \( B^\alpha \), we can find a positive constant \( C \) such that
\[
||T_g f_a||_{B^\alpha} \leq C ||f_a||_{A_p^\beta} = C,
\]
for all \( a \in \mathbb{D} \), hence for each point \( z \in \mathbb{D} \) we have
\[
(1 - |z|^2)^\alpha |f_a(z)g'(z)| \leq C.
\]
In particular, when \( z = a \), we get
\[
(1 - |a|^2)^\alpha \left( \frac{1 - |a|^2}{(1 - |a|^2)^2} \right)^{(\beta + 2)/p} |g'(a)| \leq C,
\]
whence
\[
(1 - |a|^2)^{(p\alpha - \beta - 2)/p} |g'(a)| < C.
\]
Since \( a \in \mathbb{D} \) is arbitrary, the result follows.

**Theorem 3.2.** Let \( 1 \leq p < \infty, -1 < \beta < \infty, \alpha > 0 \) and \( g : \mathbb{D} \to \mathbb{C} \) be a holomorphic map. Suppose that \( T_g \) maps \( A_p^\beta \) boundedly into \( B^\alpha \). Then \( T_g \) maps \( A_p^\beta \) compactly into \( B^\alpha \) if and only if
\[
\lim_{|z| \to 1^-} (1 - |z|^2)^{(p\alpha - \beta - 2)/p} |g'(z)| = 0. \tag{3.2}
\]
Proof. First suppose that (3.2) holds. Let \( \{f_n\} \) be a bounded sequence in \( A^p_{\beta} \) that converges to zero uniformly on compact subsets of \( \mathbb{D} \). Let \( M = \sup_n ||f_n||_{A^p_{\beta}} < \infty \). Given \( \varepsilon > 0 \), there exist an \( r \in (0, 1) \) such that if \( |z| > r \), then

\[
(1 - |z|^2)^{(p\alpha - \beta - 2)/p} |g'(z)| < \varepsilon.
\]

Thus for \( z \in \mathbb{D} \) such that \( |z| > r \), by (2.1) we have

\[
(1 - |z|^2)^{\alpha}|(T_g f_n)'(z)| = (1 - |z|^2)^{\alpha}|g'(z)||f_n(z)|
\leq (1 - |z|^2)^{\alpha - (\beta - 2)/p}|g'(z)| ||f_n||_{A^p_{\beta}}
\leq \varepsilon M,
\]

for all \( n \). On the other hand, since \( f_n \to 0 \) uniformly on compact subsets of \( \mathbb{D} \), there exist an \( n_0 \) such that if \( |z| \leq r \) and \( n \geq n_0 \), then \( |f_n'(z)| < \varepsilon \). By Theorem 3.1, we have \( g \in \mathcal{B}^\alpha \) and so we have

\[
N = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha}|g'(z)| < \infty,
\]

hence

\[
\sup_{|z| \leq r} (1 - |z|^2)^{\alpha}|(T_g f_n)'(z)| = \sup_{|z| \leq r} (1 - |z|^2)^{\alpha}|g'(z)| f_n(z)|
< \varepsilon N.
\]

The above arguments together yield

\[
||T_g f_n||_{\mathcal{B}^\alpha} = ||T_g f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha}|(T_g)' f_n(z)|
\leq \sup_{|z| \leq r} (1 - |z|^2)^{\alpha}|(T_g f_n)'(z)| + \sup_{|z| > r} (1 - |z|^2)^{\alpha}|(T_g f_n)'(z)|
\leq (N + M)\varepsilon.
\]

Thus

\[
||T_g f_n||_{\mathcal{B}^\alpha} \to 0 \text{ as } n \to \infty,
\]

hence \( T_g \) maps \( A^p_{\beta} \) compactly into \( \mathcal{B}^\alpha \).

Conversely, suppose \( T_g \) maps \( A^p_{\beta} \) compactly into \( \mathcal{B}^\alpha \) and (3.2) does not hold. Then there exists a positive number \( \delta \) and a sequence \( \{z_n\} \) in \( \mathbb{D} \) such that \( |z_n| \to 1 \) and

\[
(1 - |z|^2)^{(p\alpha - \beta - 2)/p} |g'(z_n)| \geq \delta,
\]

for all \( n \). For each \( n \), consider the function \( f_n \) defined as

\[
f_n(z) = \left( \frac{1 - |z_n|^2}{1 - \bar{z}_n z} \right)^{(\beta + 2)/p}, \quad z \in \mathbb{D}.
\]
Then the sequence \( \{f_n\} \) is norm bounded and \( f_n \to 0 \) uniformly on compact subsets of \( \mathbb{D} \), it follows that a subsequence of \( \{T_g f_n\} \) tends to 0 in \( \mathcal{B}^\alpha \). On the other hand,

\[
\|T_g f_n\|_{\mathcal{B}^\alpha} \geq (1 - |n|^2)^\alpha \|T_g f_n\|'(z_n)
\]

\[
= (1 - |n|^2)^\alpha |g'(z_n)| |f_n(z_n)|
\]

\[
= (1 - |n|^2)^{(p\alpha - \beta - 2)/p} |g'(z_n)|
\]

\[
\geq \delta,
\]

which is absurd. Hence we are done.

## 4 Riemann-Stieltjes operators between weighted Bloch spaces \( B^\alpha \)

In this section we characterize boundedness and compactness of Riemann-Stieltjes operators between weighted Bloch spaces \( B^\alpha \).

**Theorem 4.1.** Let \( \alpha > 0, \beta > 0 \) and \( g : \mathbb{D} \to \mathbb{C} \) be a holomorphic map.

(i) If \( 0 < \alpha < 1 \), then \( T_g \) maps \( B^\alpha \) boundedly into \( B^\beta \) if and only if \( g \in B^\beta \).

(ii) Operator \( T_g \) maps \( B \) boundedly into \( B^\beta \) if and only if

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(z)| \log \frac{2}{(1 - |z|^2)} < \infty.
\]

(iii) If \( \alpha > 1 \), then \( T_g \) maps \( B^\alpha \) boundedly into \( B^\beta \) if and only if

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{1+\beta-\alpha} |g'(z)| < \infty.
\]

**Proof.** First we consider the case \( \alpha > 1 \). Suppose \( T_g \) maps \( B^\alpha \) boundedly into \( B^\beta \). Consider the function

\[
f_a(z) = \frac{1 - |a|^2}{(1 - \overline{a}z)^\alpha}, \quad z \in \mathbb{D}.
\]

Then \( \|f_a\|_{B^\beta} \leq 1 + 2\alpha \). Thus \( f_a \in B^\alpha \) and \( M = \sup\{\|f_a\|_{B^\beta} : a \in \mathbb{D}\} \leq 1 + 2\alpha \). Since \( T_g \) maps \( B^\alpha \) boundedly into \( B^\beta \), we can find a positive constant \( C \) such that

\[
\|T_g f_a\|_{B^\beta} \leq C \|f_a\|_{B^\alpha} \leq CM
\]

for each \( a \in \mathbb{D} \), hence for each \( z \in \mathbb{D} \), we have

\[
(1 - |z|^2)^\beta |g'(z)| |f_a(z)| = (1 - |z|^2)^\beta \|T_g f_a\|'(z)
\]

\[
\leq CM.
\]
In particular, when $z = a$, we have
\[ (1 - |a|^2)^{1+\beta-a}|g'(a)| \leq CM. \]
Since $a \in \mathbb{D}$ is arbitrary, the result follows.

Conversely, suppose that
\[ M = \sup_{z \in D}(1 - |z|^2)^{1+\beta-a}|g'(z)| < \infty \]
By (2.3), we have
\[ |f(z) - f(0)| \leq \frac{||f||_{B^\alpha}}{(\alpha - 1)(1-|z|^2)^{(\alpha-1)}} \]
for all $z \in \mathbb{D}$, independent of $f \in B^\alpha$. Since
\[
||T_g f||_{B^\beta} = |T_g f(0)| + \sup_{z \in D}(1 - |z|^2)^{\beta}|(T_g f)'(z)| \\
\leq \sup_{z \in D}(1 - |z|^2)^{\beta}|f(z) - f(0)||g'(z)| + |f(0)| \sup_{z \in D}(1 - |z|^2)^{\beta}|g'(z)| \\
\leq \sup_{z \in D}(1 - |z|^2)^{\beta}|g'(z)| \frac{||f||_{B^\beta}}{(\alpha - 1)(1-|z|^2)^{(\alpha-1)}} + C \sup_{z \in D}(1 - |z|^2)^{\beta}|g'(z)| \\
\leq (CM + \frac{M}{(\alpha - 1)}) ||f||_{B^\beta}.
\]
Hence $T_g$ maps $B^\alpha$ boundedly into $B^\beta$. This completes the proof of (iii). Next, we will prove (ii). Suppose $T_g$ maps $B$ boundedly into $B^\beta$. For $a \in \mathbb{D}$, let
\[ f_a(z) = \log \frac{2}{(1-az)} , \quad z \in \mathbb{D}. \]
Then $f_a \in B$ and $||f_a||_B \leq 3$. So
\[
3||T_g||_{B^\beta} \geq ||T_g f_a||_{B^\beta} \\
= |T_g f_a(0)| + \sup_{z \in D}(1 - |z|^2)^{\beta}|(T_g f_a)'(z)| \\
\geq (1 - |a|^2)^{\beta}|g'(a)||f_a(a)| \\
= (1 - |a|^2)^{\beta}|g'(a)| \log \frac{2}{(1-|a|^2)}. 
\]
Since $a \in \mathbb{D}$ is arbitrary, the result follows.

Conversely, suppose that
\[ M = \sup_{z \in D}(1 - |z|^2)^{\beta}|g'(z)| \log \frac{2}{(1-|z|^2)} < \infty. \]
By (2.5), for \( f \in \mathcal{B}^\alpha \), we have
\[
\|T_g f\|_{\mathcal{B}^\beta} = |T_g f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |(T_g f)'(z)|
\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(z)| \frac{1}{\log 2} \log \frac{2}{(1 - |z|^2)} \|f\|_{\mathcal{B}^\alpha}
= \frac{1}{\log 2} M \|f\|_{\mathcal{B}^\alpha},
\]
hence \( T_g \) maps \( \mathcal{B} \) boundedly into \( \mathcal{B}^\beta \). This completes the proof of (ii). Finally we will prove (i). First suppose that \( T_g \) maps \( \mathcal{B}^\alpha \) boundedly into \( \mathcal{B}^\beta \), then
\[
g = g(0) + T_g 1 \in \mathcal{B}^\beta.
\]
Conversely, suppose that \( g \in \mathcal{B}^\beta \). Then
\[
|f(z)| \leq \|f\|_{\mathcal{B}^\alpha} (1 + (1 - |z|^2)^{1-\alpha}), \quad \alpha \neq 1, z \in \mathbb{D}.
\]
Thus, if \( f \in \mathcal{B}^\alpha \), then
\[
\|T_g f\|_{\mathcal{B}^\beta} \leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |(T_g f)'(z)|
\leq \sup_{z \in \mathbb{D}} ((1 - |z|^2)^\beta + (1 - |z|^2)^{\beta+1-\alpha}) |g'(z)| \|f\|_{\mathcal{B}^\alpha}
\leq C \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(z)| \|f\|_{\mathcal{B}^\alpha}
\leq C \|g\|_{\mathcal{B}^\beta} \|f\|_{\mathcal{B}^\alpha}.
\]
As a result, \( T_g \) maps \( \mathcal{B}^\alpha \) boundedly into \( \mathcal{B}^\beta \).

**Lemma 4.2.**[6] Let \( 0 < \alpha < 1 \) and let \( T \) be a bounded linear operator from \( \mathcal{B}^\alpha \) into a normed linear space \( X \). Then \( T \) is compact if and only if \( \|Tf_n\|_X \to 0 \), whenever \( \{f_n\} \) is a bounded sequence in \( \mathcal{B}^\alpha \) that converges to zero uniformly on \( \overline{\mathbb{D}} \).

**Theorem 4.3.** Let \( \alpha > 0, \beta > 0 \) and \( g : \mathbb{D} \to \mathbb{C} \) be a holomorphic map. Suppose that \( T_g \) maps \( \mathcal{B}^\alpha \) boundedly into \( \mathcal{B}^\beta \).

(i) If \( 0 < \alpha < 1 \), then \( T_g \) maps \( \mathcal{B}^\alpha \) compactly into \( \mathcal{B}^\beta \).

(ii) Operator \( T_g \) maps \( \mathcal{B} \) compactly into \( \mathcal{B}^\beta \) if and only if
\[
\lim_{|z| \to 1} (1 - |z|^2)^\beta |g'(z)| \log \frac{2}{(1 - |z|^2)} = 0.
\]

(iii) If \( \alpha > 1 \), then \( T_g \) maps \( \mathcal{B}^\alpha \) compactly into \( \mathcal{B}^\beta \) if and only if
\[
\lim_{|z| \to 1} (1 - |z|^2)^{1+\beta-\alpha} |g'(z)| = 0.
\]
Proof. First we consider the case \( \alpha > 1 \). To prove that the condition in (iii) is sufficient for compactness of the operator \( T_g \) from \( \mathcal{B}_\alpha \) into \( \mathcal{B}_\beta \), it is enough to show that if \( \{ f_n \} \) is a bounded sequence in \( \mathcal{B}_\alpha \) that converges to zero uniformly on compact subsets of \( D \), then \( \lim_{n \to \infty} ||T_g f_n||_{\mathcal{B}_\beta} = 0 \). Let \( M = \sup_n ||f_n||_{\mathcal{B}_\alpha} < \infty \). Given \( \varepsilon > 0 \), there exists an \( r \in (0, 1) \) such that, if \( |z| > r \), then

\[
(1 - |z|^2)^{1+\beta-\alpha} |g'(z)| < \varepsilon.
\]

By (2.3), we have

\[
|f_n(z) - f_n(0)| \leq \frac{||f_n||_{\mathcal{B}_\alpha}}{(\alpha - 1)(1 - |z|^2)^{(\alpha - 1)}}
\]

for all \( z \in D \). Since

\[
||T_g f_n||_{\mathcal{B}_\beta} = ||T_g f_n(0)| + \sup_{z \in D} (1 - |z|^2)^\beta |(T_g f_n)'(z)|
\]

\[
\leq \sup_{z \in D} (1 - |z|^2)^\beta |f_n(z) - f_n(0)||g'(z)| + |f_n(0)| \sup_{z \in D} (1 - |z|^2)^\beta |g'(z)|
\]

\[
\leq \sup_{|z| \leq \xi} (1 - |z|^2)^\beta |g'(z)||f_n(z) - f_n(0)| + |f_n(0)| \sup_{|z| \leq \xi} (1 - |z|^2)^\beta |g'(z)|
\]

\[
+ \sup_{|z| > \xi} (1 - |z|^2)^\beta |g'(z)||f_n(z) - f_n(0)| + |f_n(0)| \sup_{|z| > \xi} (1 - |z|^2)^\beta |g'(z)|
\]

\[
\leq \sup_{|z| \leq \xi} (1 - |z|^2)^\beta |g'(z)||f_n(z) - f_n(0)| + |f_n(0)| \sup_{|z| \leq \xi} (1 - |z|^2)^\beta |g'(z)|
\]

\[
+ \sup_{|z| > \xi} (1 - |z|^2)^\beta |g'(z)| |f_n(z) - f_n(0)| + |f_n(0)| \sup_{|z| > \xi} (1 - |z|^2)^\beta |g'(z)|
\]

\[
\leq \varepsilon (4||g||_{\mathcal{B}_\beta} + \frac{M}{\alpha - 1} + M) \quad \text{as} \quad n \geq n_0.
\]

Thus, \( T_g \) maps \( \mathcal{B}_\alpha \) compactly into \( \mathcal{B}_\beta \).

Conversely, suppose that \( T_g \) maps \( \mathcal{B}_\alpha \) compactly into \( \mathcal{B}_\beta \) and (iii ) does not hold. Then there exists a positive number \( \delta \) and a sequence \( \{ z_n \} \) in \( D \) such that \( |z_n| \to 1 \) and

\[
(1 - |z_n|^2)^{1+\beta-\alpha} |g'(z_n)| \geq \delta,
\]

for all \( n \). For each \( n \), let

\[
f_n(z) = \frac{1 - |z|^2}{(1 - |z| z_n)^\alpha}, \quad z \in D.
\]

Then the sequence \( f_n \) is norm bounded and \( f_n \to 0 \) uniformly on compact subsets of \( D \). Hence there exists a subsequence of \( \{ T_g f_n \} \) which tends to 0 in \( \mathcal{B}_\beta \). On the other hand,

\[
||T_g f_n||_{\mathcal{B}_\beta} \geq (1 - |z_n|^2)^\beta |(T_g f_n)'(z_n)|
\]

\[
= (1 - |z_n|^2)^\beta |g'(z_n)||f_n(z_n)|
\]

\[
= (1 - |z_n|^2)^{1+\beta-\alpha} |g'(z_n)|
\]

\[
\geq \delta,
\]
which is absurd. Hence we are done.

Next, we will prove (ii). Let \( \{f_n\} \) is a bounded sequence in \( \mathcal{B} \) that converges to zero uniformly on compact subsets of \( \mathbb{D} \). Let \( M = \sup_n \|f_n\|_\mathcal{B} < \infty \). Given \( \varepsilon > 0 \), there exists an \( r \in (0, 1) \) such that, if \( |z| > r \), then

\[
(1 - |z|^2)^\beta |g'(z)| \log \frac{2}{1 - |z|^2} < \varepsilon.
\]

By (2.5), we have

\[
|f_n(z)| \leq \frac{1}{\log 2} \|f_n\|_\mathcal{B} \log \frac{2}{1 - |z|^2}
\]

for all \( z \in \mathbb{D} \), \( f \in \mathcal{B}^\alpha \). Also, by Theorem 4.1, we have \( g \in \mathcal{B}^\beta \) and so, for the above \( \varepsilon \), we can find \( n_0 \in \mathbb{N} \) such that

\[
\|T_g f_n\|_{\mathcal{B}^\beta} = |T_g f_n(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(z)| |f_n(z)|
\]

\[
\leq \sup_{|z| \leq r} (1 - |z|^2)^\beta |g'(z)| |f_n(z)| + \sup_{|z| > r} (1 - |z|^2)^\beta |g'(z)| |f_n(z)|
\]

\[
\leq \varepsilon \left( \|g\|_{\mathcal{B}^\beta} + M \right) \quad \text{for} \quad n \geq n_0.
\]

Thus \( T_g \) maps \( \mathcal{B} \) compactly into \( \mathcal{B}^\beta \). Conversely, suppose \( T_g \) maps \( \mathcal{B} \) compactly into \( \mathcal{B}^\beta \) and condition in (ii) does not hold. Then there exists a positive number \( \delta \) and a sequence \( \{z_n\} \) in \( \mathbb{D} \) such that \( |z_n| \to 1 \) and

\[
(1 - |z_n|^2)^\beta |g'(z_n)| \log \frac{2}{1 - |z_n|^2} \geq \delta,
\]

for all \( n \). For each \( n \), let

\[
f_n(z) = \log \frac{2}{1 - z_n z}, \quad z \in \mathbb{D}.
\]

Then the sequence \( f_n \) is norm bounded and \( f_n \to 0 \) uniformly on compact subsets of \( \mathbb{D} \). By the compactness of \( T_g \) we can find a subsequence of \( \{T_g f_n\} \) which tends to 0 in \( \mathcal{B}^\beta \). On the other hand,

\[
\|T_g f_n\|_{\mathcal{B}^\beta} \geq (1 - |z_n|^2)^\beta |(T_g f_n)'(z_n)|
\]

\[
= (1 - |z_n|^2)^\beta |g'(z_n)| |f_n(z_n)|
\]

\[
= (1 - |z_n|^2)^\beta |g'(z_n)| \log \frac{2}{1 - |z_n|^2}
\]

\[
\geq \delta,
\]

which is absurd. We are done.

Finally, we will prove (i). Suppose that \( \sup_n \|f_n\|_{\mathcal{B}^\alpha} \leq M \) and \( f_n \to 0 \) uniformly on \( \overline{\mathbb{D}} \). Then

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(z)| |f_n(z)| \leq \sup_{|z| \leq 1} |f_n(z)| \|g\|_{\mathcal{B}^\beta} \to 0 \quad \text{as} \quad n \to \infty.
\]
It follows from Lemma 4.2 that $T_g$ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$.

**Lemma 4.4.** Let $1 \leq p < q < \infty$ and $-1 < \alpha < \infty$. Then the injection map from $A^p_\alpha$ into $A^q_\alpha$ is compact.

**Proof** Let $\{f_n\}_{n=1}^\infty$ be a bounded sequence in $A^q_\alpha$, and let $M = \sup_{n \in \mathbb{N}} ||f_n||_{A^q_\alpha} \leq \infty$. By 2.1, $\{f_n : n \in \mathbb{N}\}$ is a normal family, hence we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges uniformly on compact subsets of $\mathbb{D}$ to an analytic function $f$. By Fatou’s lemma,

$$\int_{\mathbb{D}} |f(z)|^p \nu_\alpha(z) \leq \liminf_{k \to \infty} \int_{\mathbb{D}} |f_{n_k}(z)|^p \nu_\alpha(z) \leq M^q < \infty.$$ 

Since $p < q$, it follows that $f \in A^q_\alpha$. We claim that $\{f_{n_k}\}$ converges to $f$ in $A^p_\alpha$. Let $\varepsilon > 0$ and let $\Omega$ be an arbitrary compact subset of $\mathbb{D}$. Now

$$\int_{\mathbb{D} \setminus \Omega} |(f_{n_k} - f)(z)|^p \nu_\alpha(z) \leq \left( \int_{\mathbb{D} \setminus \Omega} |(f_{n_k} - f)(z)|^q \nu_\alpha(z) \right)^{p/q} \left( \int_{\mathbb{D} \setminus \Omega} \nu_\alpha(z) \right)^{1 - p/q}$$

$$\leq \left( \int_{\mathbb{D}} |(f_{n_k} - f)(z)|^q \nu_\alpha(z) \right)^{p/q} (\nu_\alpha(\mathbb{D} \setminus \Omega))^{1 - p/q}$$

$$\leq \left( 2^q \int_{\mathbb{D}} |(f_{n_k}(z)|^q + |f(z)|^q \nu_\alpha(z) \right)^{p/q} (\nu_\alpha(\mathbb{D} \setminus \Omega))^{1 - p/q}$$

$$\leq 2^q (||f_{n_k}||_{A^q_\alpha}^{p/q} + ||f||_{A^q_\alpha}^{p/q}) (\nu_\alpha(\mathbb{D} \setminus \Omega))^{1 - p/q}$$

$$\leq 2^q + M^p (\nu_\alpha(\mathbb{D} \setminus \Omega))^{1 - p/q},$$

where in the first line we have used Holder’s inequality and the elementary inequalities

$$(x + y)^a \leq 2^a (x^a + y^b), \quad (x + y)^b \leq (x^b + y^b)$$

which holds when $x, y \geq 0$, and $0 < b < 1$. By choosing the compact set $\Omega$ so that $\mathbb{D} \setminus \Omega$ has sufficiently small area, we obtain

$$\int_{\mathbb{D} \setminus \Omega} |(f_{n_k} - f)(z)|^p \nu_\alpha(z) < \varepsilon/2$$

for $k$ large enough. On the other hand, since $f_n \to f$ uniformly on $\Omega$ we can choose $k$ large enough so that

$$\int_{\Omega} |(f_{n_k} - f)(z)|^p \nu_\alpha(z) < \varepsilon/2.$$ 

Thus $\{f_{n_k}\}$ converges to $f$ in $A^p_\alpha$. Hence the injection map is compact.

**Corollary 4.5.** Let $1 \leq p < \infty$, $-1 < \alpha < \infty$, $\alpha > 0$ and $q : \mathbb{D} \to \mathbb{C}$ be holomorphic. Then $T_g$ maps $\mathcal{B}^\alpha$ compactly into $A^p_\beta$ if and only if $T_g$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.

**Proof.** Suppose $T_g$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, thus also into the large space
Since convergence in either space implies uniform convergence on compact sets, it follows from closed graph theorem that $T_g$ maps $B^\alpha$ boundedly into $A^p_\beta$. In order to show that $T_g$ maps $B^\alpha$ compactly into $A^p_\beta$, choose any $q$ such that $q > p$ and factorize $T_g$ through the intermediate space $A^q_\beta$:

$$B^\alpha \xrightarrow{T_g} A^q_\beta \xrightarrow{I} A^p_\beta,$$

where $\tilde{T}_g$ is the Riemann-Stieltjes operator from $B^\alpha$ to $A^q_\beta$, and $I$ is the injection map. Since $I$ is compact and $\tilde{T}_g$ is bounded, so $T_g$ maps $B^\alpha$ compactly into $A^p_\beta$.

References


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