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σ-prime rings with a special kind of automorphism

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Abstract

In this paper we investigate the relationship between the primeness and the $\sigma-$primeness for a ring with involution σ. Furthermore, if a $\sigma-$prime ring R has an automorphism $f \neq 1$ satisfying $f(x)-x$ is zero or invertible for all x in a nonzero σ-ideal I and if σ commutes with f on I, then R is either simple or $R = D \oplus \sigma(D)$ for some division ring D.

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1 Introduction

An involution on a ring R is a map $\sigma : R \longrightarrow R$ which satisfies: $\sigma(x + y) = \sigma(x) + \sigma(y)$, $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$, for all $x, y \in R$. There has been considerable interest in connection between the structure and the $\sigma-$structure of a ring with involution σ. In [5], authors defined a new class of rings with involution called σ-semisimple rings. Moreover, for left artinian rings they showed that the $\sigma-$semisimplicity coïncides with the semisimplicity. In [3], connections permitting transfer of various properties from the $\sigma-$ideal structure to the ideal structure or vice-versa, are obtained.

In the second section of this paper we establish some properties of $\sigma-$prime rings. A $\sigma-$primeness criteria for a ring with involution (R, σ) is then obtained in term of $Sa_\sigma(R)$. Furthermore, conditions under which a $\sigma-$prime ring becomes a prime ring are presented. The third section of the present paper is devoted to examine the structure of a $\sigma-$prime ring, which is not prime, provided with a particular automorphism f satisfying $f(x)-x$ is either zero or invertible, for every x in a nonzero $\sigma-$ideal of R. Examples are given illustrating some of the found results.

For clarity, it is interesting to elucidate some of the terminology to be used. Let R be a ring with unity 1, throughout this paper R^0 will always denote the opposite ring of R. If R is provided with an involution σ, then an ideal I of R is called a $\sigma-$ideal if $\sigma(I) \subseteq I$. We say R is $\sigma-$simple if R has no nonzero proper $\sigma-$ideals. A subset X of R is $\sigma-$essential in R if X has nonzero and non-empty intersection with every nonzero $\sigma-$ideal of R. An ideal P of R is said to be a $\sigma-$prime ideal if $IJ \subseteq P\,(a, b \in R$ such that $aRb \subseteq P$ and $aR\sigma(b) \subseteq P)$ implies $I \subseteq P$ or $J \subseteq P\,(a \in P$ or $b \in P)$, where I and J are $\sigma-$ideals of R. Thus R is a $\sigma-$prime ring if the zero ideal is $\sigma-$prime. If R is a non-prime ring, then R is $\sigma-$prime *if and only if* there exists a nonzero prime ideal P of R which satisfies $P \cap \sigma(P) = (0)$ ([2], Theorem 4.2). Recall that an ideal P is called prime if and only if $aRb \subseteq P$ implies $a \in P$ or $b \in P$, for any a, $b \in R$. An element x in R is said to be symmetric (resp. skew-symmetric) if x is invariant under σ (resp. $\sigma(x) = -x$). In all that follows, $Sa_\sigma(R)$ will stand for the set of all symmetric and skew-symmetric elements of R and $\mathrm{inv}(\sigma)$ will stand for the set of all central symmetric elements of R. Finally, an homomorphism f of R is said to be a $\sigma-$homomorphism if σ commutes with f (i.e. $f \circ \sigma = \sigma \circ f$).

2 $\sigma-$prime rings

Our aim in this section is to investigate some properties of $\sigma-$prime rings. For clarity, it is worthwhile to observe that a prime ring with an involution σ is σ-prime; however the converse is in general not true.
In the following proposition, we use $Sa_\sigma(R)$ to give a sufficient condition under which R becomes a $\sigma-$prime ring.

Proposition 2.1 *Let (R, σ) be a ring with involution. If every nonzero element in $Sa_\sigma(R)$ is regular then R is a $\sigma-$prime ring.*

Proof. Let I and J be σ-ideals of R such that $IJ = (0)$ with $I \neq (0)$. To prove that $J = (0)$ it then suffices to show that I contains a nonzero symmetric element. Indeed, if $0 \neq y \in I$ satisfies $\sigma(y) = y$ then $yj = 0$ for all $j \in J$ and therefore $j = 0$ proving $J = (0)$. Accordingly R is a $\sigma-$prime ring.

For existence of a nonzero symmetric element in I, as $\sigma(x)x$ is invariant under σ it then suffices to show that $\sigma(x)x \neq 0$ for some $x \in I$. If $\sigma(x)x = 0$ for all $x \in I$, since $\sigma(x) - x$ in I then $\sigma(\sigma(x) - x)(\sigma(x) - x) = -(\sigma(x) - x)^2 = 0$. As $\sigma(x) - x$ in $Sa_\sigma(R)$, thus $\sigma(x) - x = 0$ in such a way that $\sigma(x) = x$ and then x in $Sa_\sigma(R)$. Using $0 = \sigma(x)x = x^2$ together with x in $Sa_\sigma(R)$, we then get $x = 0$ which contradicts $I \neq (0)$. ∎

The converse of Proposition 2.1 is not true. Indeed, let R be the matrix algebra $M_2(K)$ over a field K and σ be the transpose involution on R. It is obvious that (R, σ) is a $\sigma-$prime ring, but the nonzero symmetric element $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not regular.

We now present example of a ring satisfying hypothesis of Proposition 2.1.

Example 1. Let $R = D \times D^0$ where D is a division ring and let σ be the exchange involution defined by $\sigma(x, y) = (y, x)$. Since nonzero symmetric (resp. skew-symmetric) elements of R are (a, a) (resp. $(a, -a)$) where $0 \neq a$ in D, these elements are clearly regular.

To study the relationship between the primeness and the $\sigma-$primeness recall that σ is said to be *anisotropic* if $\sigma(x)x = 0 \Rightarrow x = 0$: for all $x \in R$.

In the following theorem we give some properties of a $\sigma-$prime ring and we establish a necessary and sufficient condition under which a $\sigma-$prime ring becomes a prime ring.

Theorem 2.2 *For a $\sigma-$ prime ring R, the following statements hold:*
(1) If σ restricts to a certain $\sigma-$ideal is anisotropic then σ is anisotropic.
(2) $I \cap Sa_\sigma(R) \neq 0$ for every nonzero $\sigma-$ideal I of R.
(3) If $0 \neq I$ is a $\sigma-$ideal and $aIb = 0 = aI\sigma(b)$, then $a = 0$ or $b = 0$.
(4) R is a prime ring if and only if $\sigma(I)I \neq 0$ for every nonzero ideal I of R.

Proof. (1) Let I be a $\sigma-$ideal of R such that σ/I is anisotropic. If $r \in R$ satisfies $\sigma(r)r = 0$ then $\sigma(rx)rx = 0$ for all $x \in I$. The fact that $rx \in I$ yields $rx = 0$ so $rI = 0$, hence $rRx = 0$. Since I is a $\sigma-$ideal one also gets $rR\sigma(x) = 0$. The $\sigma-$primeness of R implies $r = 0$ and therefore σ is anisotropic.
(2) Suppose $I \cap Sa_\sigma(R) = 0$, then for any $x \in I$ we must have $xSa_\sigma(R)\sigma(x) \subset I \cap Sa_\sigma(R) = 0$. Hence, for $r \in R,\ x(\sigma(r) - r)\sigma(x) = 0$ so $x\sigma(r)\sigma(x) = xr\sigma(x)$ which implies that $xr\sigma(x) \in I \cap Sa_\sigma(R)$, consequently $xR\sigma(x) = 0$. Since $\sigma(x) + x \in I \cap Sa_\sigma(R)$ then $\sigma(x) = -x$ and the $\sigma-$primeness of R yields $x = 0$. Therefore, if $I \neq 0$ we may conclude $I \cap Sa_\sigma(R) \neq 0$.
(3) Suppose $a \neq 0$, there exists some $x \in I$ such that $ax \neq 0$. Indeed, otherwise

$aRx = 0$ and $aR\sigma(x) = 0$ for all $x \in I$ so $a = 0$. Since $aIRb = 0$ and $aIR\sigma(b) = 0$, in particular $axRb = axR\sigma(b) = 0$ which implies that $b = 0$.
(4) Let I and J be ideals of R such that $IJ = 0$. Setting $L = \sigma(I)I$ and $K = J\sigma(J)$, clearly L and K are $\sigma-$ideals of R which satisfies $LK = 0$. Since R is $\sigma-$prime then $L = 0$ or $K = 0$ so $I = 0$ or $J = 0$ proving the primeness of R. The converse is trivial. ∎

Corollary 2.3 *Let R be a σ-prime ring. If σ is anisotropic then R is prime.*

For (1) of Theorem 2.2, the $\sigma-$primeness of R is necessary. Indeed, in the following example we construct a non σ-prime ring with isotropic involution having a nonzero $\sigma-$ideal on which σ induces an anisotropic involution.

Example 2. Let S_3 be the symmetric group on 3 letters, then $S_3 = \{1, t, tc, tc^{-1}, c, c^{-1}\}$, where $t^2 = (tc)^2 = (tc^{-1})^2 = 1 = c^3$ and $tct = c^{-1}$. Consider the group ring $\mathbb{C}[S_3]$ provided with its canonical involution defined by $\sigma(\sum_g a_g g) = \sum_g a_g g^{-1}$. According to [[4], p. 140], if $e_1 = \frac{1}{6}\sum_g g$, $e_2 = \frac{1}{6}(1 - t - tc - tc^{-1} + c + c^{-1})$ and $e_3 = \frac{2}{6}(2 - c - c^{-1})$ then $\mathbb{C}[S_3] = \oplus_{i=1}^{3} A_i$ where $A_1 = \mathbb{C}e_1 \cong \mathbb{C}$, $A_2 = \mathbb{C}[S_3]e_2$ and $A_3 = \mathbb{C}[S_3]e_3$. Clearly A_1 is a $\sigma-$ideal on which σ induces an anisotropic involution. Set $x = \frac{i}{\sqrt{3}}(1 + 2c)e_3 \in A_3$, then $\sigma(x) = -x$ so $e = \frac{1}{2}(e_3 + x)$ is a nonzero element of A_3 satisfying $\sigma(e)e = 0$ proving that σ is isotropic.

Now, we collect some properties of a $\sigma-$prime ring which is not prime.

Theorem 2.4 *Let R be a $\sigma-$prime ring which is not prime, then:*
(1) Every nonzero central element in $Sa_\sigma(R)$ is regular.
(2) If $I,\ J \subset R$ are nonzero $\sigma-$ideals then $I \cap J$ is a nonzero $\sigma-$ideal of R.
(3) If $Z(R)$ is $\sigma-$essential in R then $inv(\sigma)$ is $\sigma-$essential in R.

Proof. (1) Since R is not prime there exists a nonzero prime ideal P of R with $P \cap \sigma(P) = (0)$. Let $0 \neq x \in Sa_\sigma(R)$ and let $r \in R$ such that $xr = 0$. If x is in the center of R, then $xRr = 0$. Since $\sigma(x) = \pm x$ necessarily $x \notin P$. If not, $x \in P \cap \sigma(P)$ and thus $x = 0$. Using the primeness of P we get $r \in P$. Similarly, the fact that $xr \in \sigma(P)$ yields $r \in \sigma(P)$ so $r = 0$.
(2) Assume for purposes of contradiction that $I \cap J = (0)$ then $IJ \subset P \cap \sigma(P)$. As I is a $\sigma-$ideal, if $I \subset P$ then $I \subset P \cap \sigma(P)$ and thus $I = (0)$; hence $I \not\subset P$. A similar reasoning shows that $J \not\subset P$, which contradicts the primeness of P. Consequently, $I \cap J \neq (0)$.
(3) Suppose $Z(R)$ is $\sigma-$essential in R and let I be a nonzero $\sigma-$ideal of R. There exists $0 \neq x$ a central element in I and either $\sigma(x)x \neq 0$ or $\sigma(x)+x \neq 0$. For if not, $\sigma(x) = -x$ so x is a central element in $Sa_\sigma(R)$. In view of (1), x is regular and $\sigma(x)x = 0$ which contradicts $x \neq 0$. As $\sigma(x)x$ and $\sigma(x) + x$ are

both central symmetric elements this completes our proof. ∎

Remark. (1) of Theorem 2.4 shows that the converse of Proposition 2.1 is true for central elements in $Sa_\sigma(R)$.

3 Automorphisms in $\sigma-$prime rings

Our aim in this section is to investigate the structure of a $\sigma-$prime ring, which is not prime, having a special kind of automorphism. More precisely, we shall prove that under suitable assumptions $R = D \oplus \sigma(D)$ for a division ring D.

Throughout this section, *R will always be a $\sigma-$prime ring which is not prime and $f \neq 1$ will be an automorphism of R such that $f(x) - x$ is either 0 or invertible, for every element x in a nonzero $\sigma-$ideal I of R.*

Lemma 3.1 *Let J be a nonzero σ-ideal of R and let g be an homomorphism of R. If $g(x) = x$ for all x in J, then g is the identity on R.*

Proof. Let r in R, for every x in J we have $g(rx) = g(r)x = rx$, so that $(g(r) - r)x = 0$. Accordingly, $(g(r) - r)Rx = 0$ and the fact that J is a $\sigma-$ideal gives $(g(r) - r)R\sigma(x) = 0$. As R is σ-prime we then get $g(r) = r$ for all r in R. ∎

Remark. It is worthwhile to note that a σ-prime ring is in general not σ-simple. Indeed, let A be a prime ring which is not simple and let σ be the exchange involution defined on $R = A \times A^o$. If T is a nonzero proper ideal of A then $T \times T$ is a nonzero proper $\sigma-$ideal of R so that R is not a $\sigma-$simple ring. For the $\sigma-$primeness of R, let L and K be $\sigma-$ideals of R such that $LK = (0)$. Hence $L = N \times N$ and $K = J \times J$ where N and J are ideals of A such that $NJ = 0$. Since A is prime we obtain $N = (0)$ or $J = (0)$. Therefore $L = (0)$ or $K = (0)$ in such a way that R is a $\sigma-$prime ring.

Lemma 3.2 *If σ commutes with f on I, then R is a $\sigma-$simple ring.*

Proof. Let $0 \neq K$ be a $\sigma-$ideal of R and setting $L = I \cap K$. From Theorem 2.4 it follows that $L \cap f(L)$ is a nonzero $\sigma-$ideal of R contained in I. According to Lemma 3.1, there is x in $L \cap f(L)$ such that $f(x) - x$ is invertible. As $f(x) - x$ in $f(L)$, then $f(L) = R$. Thus, $L = R$ and then $K = R$ proving our theorem. ∎

Consequence. The unique σ-ideal I such that σ commutes with f on I is R.

Now we construct a $\sigma-$prime ring satisfying hypothesis of Lemma 3.2.
Example 3. Let (D,τ) be a division ring with involution and let $R = D\times D$. The map f defined by $f(x,y) = (y,x)$ is obviously a nontrivial automorphism of R. Moreover, if σ is the natural extension of τ to R, then σ commutes with f and $f(x,y)-(x,y)$ is either zero or invertible for all (x,y) in R.

Lemma 3.3 *If f is a σ-automorphism and if R is not simple, then there exists a central idempotent e in R such that $1 = e+\sigma(e)$. Moreover, if e' is a central idempotent satisfying $1 = e'+\sigma(e')$ then $e' = e$ or $e' = \sigma(e)$.*

Proof. Let N be a nonzero proper ideal of R. Since σ commutes with f, in view of Lemma 3.2, R is then $\sigma-$simple and thus $R = N\oplus\sigma(N)$. Hence $1 = e+\alpha$ where $e\in N$ and $\alpha\in\sigma(N)$. More precisely, e and α are central idempotents such that $e\alpha = 0$. As $\sigma(1) = 1$, then $e+\alpha = \sigma(e)+\sigma(\alpha)$ so $e-\sigma(\alpha) = \sigma(e)-\alpha\in N\cap\sigma(N)$ proving $\alpha = \sigma(e)$; consequently $1 = e+\sigma(e)$. For the uniqueness of e, suppose $1 = e'+\sigma(e')$ with e' an idempotent in $Z(R)$. If $e\neq e'$, then $\mu = e'-e$ is a central element such that $\sigma(\mu) = -\mu$. Since $R\mu$ is a nonzero $\sigma-$ideal of R we then get $R\mu = R$. If r in R is such that $r\mu = 1$, then $r\mu = \sigma(r)\sigma(\mu) = -\sigma(r)\mu$ and thus $(\sigma(r)+r)\mu = 0$. As μ is a central element in $Sa_\sigma(R)$, (1) of Theorem 2.4 yields $\sigma(r) = -r$. In view of $1 = re'-re$ we deduce $e' = re'-ree' = re'\sigma(e)$ and using $e = re'e-re = -re\sigma(e')$ we obtain $\sigma(e) = r\sigma(e)e'$ so $e' = \sigma(e)$. ∎

In [[3], Proposition 2.1] it is proved, by using $\sigma-$ideals, that every $\sigma-$simple ring R which is not simple can be decomposed as $R = B\oplus\sigma(B)$ where B is a simple subring of R. In our case, we can give a short proof for this result by using idempotents elements and Lemma 3.3.

Proposition 3.4 *If f is a $\sigma-$automorphism of R, then R is either simple or $R = B\oplus\sigma(B)$ where B is a simple subring of R.*

Proof. If R is not simple, then we have $R = Re\oplus R\sigma(e)$ by Lemma 3.3. If $0\neq J$ is an ideal of Re then $R = J\oplus\sigma(J)$. Thus $J = Re'$ where e' is a central idempotent in R satisfying $1 = e'+\sigma(e')$. Once again using Lemma 3.3 we conclude $e' = e$ or $e' = \sigma(e)$. Since $e'\in J\subset Re$ and $\sigma(e)\notin Re$, then $e' = e$ proving the simplicity of Re. ∎

Lemma 3.5 *If x in R satisfies $f(x)-x = 0$, then $x = 0$ or x is invertible.*

Proof. Since $f\neq 1$, Lemma 3.1 assures existence of an element z in I such that $f(z)-z\neq 0$ and thus $f(z)-z$ is invertible. If $x\neq 0$, the fact that $f(zx)-zx = (f(z)-z)x$ together with zx in I implies $f(zx)-zx$ is invertible, proving the invertibility of x. ∎

Remark. In [1], it is proved that if R is a ring having an automorphism $\Phi \neq 1$ such that $\Phi(r) - r$ is 0 or invertible for every r in R, then R is simple or $R = D \oplus \Phi(D)$ where D is a division ring. In our case, by combining Lemma 3.3 and Lemma 3.5, we give an additional description of R as follows.

Theorem 3.6 *Let R be a σ-prime ring which is not prime having an automorphism $f \neq 1$ such that $f(x) - x$ is 0 or invertible for every x in a nonzero σ-ideal I of R. If σ commutes with f on I, then R is either simple or $R = D \oplus \sigma(D)$, where D is a division ring.*

Proof. In view of Lemma 3.2, R is a σ-simple ring. Hence $I = R$ and thus f is a σ-automorphism and $f(r) - r$ is 0 or invertible for all r in R. Now if R is not simple, we have seen above that $R = B \oplus \sigma(B)$ where $B = Re$. Since

$$f(1) = 1 = e + \sigma(e) = f(e) + f(\sigma(e)) = f(e) + \sigma(f(e))$$

by view of Lemma 3.3 we deduce $f(e) = e$ or $f(e) = \sigma(e)$. If $f(e) = e$, then by virtue of Lemma 3.5 e is zero or invertible which is impossible. Thus $f(e) = \sigma(e)$ so $f(B) = \sigma(B)$. Accordingly, $R = B \oplus f(B)$ and $Bf(B) = 0$. Let x be a nonzero element in B; if $f(x) - x$ is zero then x is invertible in R by Lemma 3.5. Then there is r in R such that $xr = 1_R = e + \sigma(e)$. Hence $x(re) = e$, so that x is invertible in B. If $f(x) - x$ is invertible in R, then $(f(x) - x)r = 1 = e + \sigma(e)$, for some r in R. Since $f(x)re$ in $f(B)B = 0$, hence $-xre = x(-re) = e$ proving that x in invertible in B. Consequently, B is a division ring. ∎

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