Modules which Satisfy the Radical Formula

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Abstract

Let R be a commutative ring with non-zero identity. Our aim is to investigate MacCasland modules. A number of results concerning MacCasland modules are given. For example, it is shown that every prime representable module over a commutative ring is MacCasland.

Mathematics Subject Classification: 13A15, 13F05, 13A10

Keywords: Prime, MacCasland

1 Introduction

In this paper all rings are commutative rings with non-zero identity and all modules are unital. Recently, extensive research has been done on prime submodules, see, for examples [1, 4, 7]. In this paper, we continue these investigations. A proper submodule N of M is prime if for any $r \in R$ and $m \in M$ such that $rm \in N$, either $m \in N$ or $r \in (N : M) = \{a \in R : aM \subseteq N\}$. It is easy to show that if N is a prime submodule of M then the annihilator P of the module M/N is a prime ideal of R , and N is said to be P -prime submodule of M .

Let N be a proper submodule of an R -module M . The radical of N in M , denoted by $\text{rad}_M(N)$, is defined to be the intersection of all prime submodules of M containing N . Should there be no prime submodule of M containing N , then we put $\text{rad}_M(N) = M$. The envelope submodule $\text{RE}_M(N)$ of N in M is a submodule of M generated by the set $\text{E}_M(N) = \{rm : r \in R \text{ and } m \in M \text{ such that } r^n m \in N \text{ for some } n \in \mathbf{N}\}$.

An R -module $M \neq 0$ is called a secondary module provided that for every element $r \in R$, the R -endomorphism of M produced by multiplication by r is either surjective or nilpotent. This implies that $\text{nilrad}(M) = P$ is a prime ideal of R , and M is said to be P -secondary. A secondary representation for

an R -module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable (see [6]).

We return now to the radical of an R -submodule N of M , in order to discuss the connection between $\text{rad}_M(N)$ and the envelope of N , as defined in the above. We will call from [1] N a McCasland submodule in M if it satisfies the radical formula, that is, $\text{RE}_M(N) = \text{rad}_M(N)$. Likewise, M will be called a McCasland module if every submodule of M is a McCasland submodule. We shortly summarize the content of the paper. In section 2, Theorem 2.3, it is shown that every prime representable module over a commutative ring is MacCasland. In theorem 2.7, it is shown that every finitely generated second module over a commutative ring is MacCasland. Finally, assume that $R = R_1 \times R_2$ where each R_i is a commutative ring and let $M = M_1 \times M_2$ be an R -module. It is shown that, in Theorem 2.13, M is MacCasland R -module if and only if M_i is a MacCasland R_i -module for $i = 1, 2$.

2 Prime submodules

An R -module M is said to be prime if the zero submodule of M is a prime submodule of M , so M is prime if and only if whenever $rm = 0$ either $m = 0$ or $rM = 0$. A non-zero submodule N of M is said to be second submodule of M if for each $r \in R$, the R -endomorphism of N produced by multiplication by r is either surjective or zero. We say M is a second module if M is a second submodule of itself [8]. Our starting point is the following lemma:

Lemma 2.1 *Assume that M is a module over a commutative ring R and let I be an ideal of R such that $I \subseteq (0 : M)$. Then M is a MacCasland R -module if and only if M is MacCasland as an R/I -module.*

Proof. Let N be a submodule of M . Clearly, $\text{rad}_M(N) = \text{RE}_M(N)$ as an R -module if and only if $\text{rad}_M(N) = \text{RE}_M(N)$ as an R/I -module. $\square$

Proposition 2.2 *Let M be a representable module over a commutative ring R such that 0 is a prime submodule. Then M is a second R -module.*

Proof. Let $M = \sum_{i=1}^n M_i$ where for each $1 \leq i \leq n$, M_i is P_i -secondary. For each $1 \leq i \leq n$, let $m_i \in M_i$ and $p_i \in P_i$, so $p_i^{s_i} m_i = 0$ for some s_i . Then 0 prime gives either $P_i \subseteq (0 : M) = P$ or $M_i = 0$; hence we must have $P_i \subseteq P$ for all $1 \leq i \leq n$. It follows from [6, Theorem 2.3] that for each i , $P_i \subseteq P \subseteq \bigcap_{j=1}^n P_j \subseteq P_i$, so M is finite sum of P -second modules; hence it is P -second by [8, Proposition 3.2]. $\square$

Theorem 2.3 *If M is a representable module over a commutative ring R such that 0 is a prime submodule, then M is a MacCasland R -module.*

Proof. By Proposition 2.2 and [8, Theorem 1.3], M is a divisible $R/(0 : M)$ -module, so it is macCasland as an $R/(0 : M)$ -module by [7, Theorem 2.2]. Now the assertion follows from Lemma 2.1. $\square$

Corollary 2.4 *If M is an Artinian module over a commutative ring R such that 0 is a prime submodule, then M is a MacCasland R -module.*

Proof. This follows from Theorem 2.3. $\square$

We call a non-zero submodule N of M , a minimal submodule, if it is not strictly contains any other non-zero submodule of M [8].

Theorem 2.5 *Assume that R is a commutative ring and let N be a minimal R -submodule of M . Then N is MacCasland.*

Proof. By [8, Proposition 1.6], N is P -second where P is a maximal ideal of R , so $N/\{0\}$ is torsion-free as an R/P -module; hence 0 is a prime submodule of N . Now the assertion follows from Theorem 2.3. $\square$

Theorem 2.6 *Let P be a prime ideal of a commutative ring R , I an index set and M a P -second R -module such that 0 is a prime submodule of $M^{(I)}$. Then $M^{(I)}$ is MacCasland.*

Proof. This follows from [2, Proposition 3.3] and Theorem 2.3. $\square$

Theorem 2.7 *Every finitely generated second module over a commutative ring R is MacCasland.*

Proof. By [8, Theorem 3.5], 0 is a prime submodule of M . Now the assertion follows from Theorem 2.2. $\square$

Let $R = R_1 \times R_2$ where each R_i is a commutative ring with non-zero identity. Then an ideal $I = I_1 \times I_2$ of R is prime (resp. Maximal) if and only if I_1, I_2 is equal to the corresponding ring R_1, R_2 , and the other is prime (resp. maximal). Moreover, any R -module M can be uniquely decomposed into a direct product of modules, i.e. $M = M_1 \times M_2$ where $M_1 = (1, 0)M$ is an R_1 -module and $M_2 = (0, 1)M$ is an R_2 -module.

Proposition 2.8 *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with non-zero identity. Let M_i be an R_i -module and let $M = M_1 \times M_2$ be the R -module with action $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ where $r_i \in R_i$ and $m_i \in M_i$. Then the following hold:*

(i) *M is a $(P \times R_2)$ -prime R -module if and only if $M_2 = 0$ and M_1 is a P -prime R_1 -module.*

(ii) M is a $(R_1 \times Q)$ -prime R -module if and only if $M_1 = 0$ and M_2 is a Q -prime R_2 -module.

(iii) M is a $(P \times R_2)$ -secondary R -module if and only if $M_2 = 0$ and M_1 is a P -secondary R_1 -module.

(iv) M is a $(R_1 \times Q)$ -secondary R -module if and only if $M_1 = 0$ and M_2 is a Q -secondary R_2 -module.

Proof. (i) Assume that M is a $(P \times R_2)$ -prime R -module and let $r_1 m_1 = 0$ where $r_1 \in R_1$ and $m_1 \in M_1$. Then $(r_1, 0)(m_1, 0) = 0$, so M prime gives either $m_1 = 0$ or $(P \times R_2)M_1 \subseteq (P \times R_2)M = 0$; hence either $m_1 = 0$ or $PM_1 = 0$. Thus M_1 is P -prime. Let $x \in M_2$ and $a \notin R_1 - P$. Then $(a, 0)(0, x) = 0$, so $x = 0$ since $(a, 0) \notin P \times R_2$. Therefore, $M_2 = 0$. Conversely, assume that $M_2 = 0$ and M_1 is P -prime. Let $(r_1, r_2)(b, 0) = 0$. Then M_1 prime gives either $b = 0$ or $r_1 \in P$, as required.

(ii) This proof is similar to that in case (i) and we omit it.

(iii) Assume that M is a $(P \times R_2)$ -secondary R -module and let $a \in R_1$. Then $(a, 1) \in P \times R_2$, so $(a^n, 1)M = 0$ for some n ; hence $a^n M_1 = 0$ and $M_2 = 0$. If $a \notin P$, then $(a, 0)M = M$, so $aM_1 = M_1$. Thus, M_1 is P -secondary.

(iv) This proof is similar to that in case (iii) and we omit it. $\square$

Lemma 2.9 *Let the situation be as in (2.8), and let $N = N_1 \times N_2$ be a submodule of M . Then the following hold:*

(i) N is a $(P \times R_2)$ -prime R -submodule of M if and only if $M_2 = N_2$ and N_1 is a P -prime R_1 -submodule of M_1 .

(ii) N is a $(R_1 \times Q)$ -prime R -submodule of M if and only if $M_1 = N_1$ and N_2 is a Q -prime R_2 -submodule of M_2 .

Proof. (i) By [3, Theorem 2.14], it is enough to show that if N is a $(P \times R_2)$ -prime R -submodule of M , then $M_2 = N_2$ and N_1 is a P -prime R_1 -submodule of M_1 . Let $ax \in N_1$ where $a \in R_1$ and $x \in M_1$. Then $(a, 1)(x, 0) \in N$, then N prime gives either $(a, 1) \in (N : M) = (N_1, M_1) \times (N_2 : M_2)$ or $(x, 0) \in N$, so either $a \in (N_1 : M_1)$ and $M_2 = N_2$ or $x \in N_1$, as needed.

(ii) This proof is similar to that in case (i) and we omit it. $\square$

Lemma 2.10 *Let the situation be as in (2.8), and let $N = N_1 \times N_2$ be a submodule of M . Then the following hold:*

(i) $\text{rad}_M(N_1 \times M_2) = \text{rad}_{M_1}(N_1) \times M_2$.

(ii) $\text{rad}_M(M_1 \times N_2) = M_1 \times \text{rad}_{M_2}(N_2)$.

Proof. This follows from Lemma 2.9. $\square$

Proposition 2.11 *Let the situation be as in (2.8), and let $N = N_1 \times N_2$ be a submodule of M . Then*

$$\text{rad}_M(N) = (\text{rad}_{M_1}(N_1) \times M_2) \cap (M_1 \times \text{rad}_{M_2}(N_2)).$$

Proof. Using Lemma 2.10, we see that $\text{rad}_M(N) =$

$$\text{rad}_M(N_1 \times M_2) \cap \text{rad}_M(M_1 \times N_2) = (\text{rad}_{M_1}(N_1) \times M_2) \cap (M_1 \times \text{rad}_{M_2}(N_2)).$$

□

Proposition 2.12 *Let the situation be as in (2.8), and let $N = N_1 \times N_2$ be a submodule of M . Then $\text{RE}_M(N) = \text{RE}_{M_1}(N_1) \times \text{RE}_{M_2}(N_2)$.*

Proof. $x = \sum_{i=1}^k (r_i, s_i)(x_i, y_i) \in \text{RE}_M(N)$ where $(r_i^{n_i}, s_i^{n_i})(x_i, y_i) \in N$ for some n_i if and only if $u = \sum_{i=1}^k r_i x_i \in N_1$ with $r_i^{n_i} x_i \in N_1$ and $w = \sum_{i=1}^k s_i y_i \in N_2$ with $s_i^{n_i} y_i \in N_2$. Then $x = (u, w) \in \text{RE}_M(N)$ if and only if $u \in \text{RE}_{M_1}(N_1)$ and $w \in \text{RE}_{M_2}(N_2)$, as required. □

Theorem 2.13 *Let the situation be as in (2.8). M is a MacCaslan R -module if and only if M_i is a MacCaslan R_i -module for $i = 1, 2$.*

Proof. This follows from Proposition 2.11 and Proposition 2.12. □

Theorem 2.14 *Let the situation be as in (2.8), and let M be a second R -module. Then the following hold:*

- (i) *If M is $(P \times R_2)$ -prime, then M is MacCaslan if and only if M_1 is a MacCaslan R_1 -module.*
- (ii) *If M is $(R_1 \times Q)$ -prime, then M is MacCaslan if and only if M_2 is a MacCaslan R_2 -module.*

Proof. (i) Since M is $(P \times R_2)$ -prime, we must have M is $(P \times R_2)$ -second (otherwise $M = 0$ by Proposition 2.8). Now the assertion follows from Proposition 2.8, Theorem 2.13 and Theorem 2.3.

(ii) This proof is similar to that in case (i) and we omit it. □

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Received: June 12, 2006