

Derivations and commutativity of σ -prime rings

L. Oukhtite and S. Salhi

Université Moulay Ismaïl, Faculté des Sciences et Techniques
Département de Mathématiques, Groupe d'Algèbre et Applications
B. P. 509 Boutalamine, Errachidia; Maroc.
oukhtite@math.net

Abstract

Let R be a σ -prime ring with characteristic not two and d be a nonzero derivation of R commuting with σ . The purpose of this paper is to give suitable conditions under which R must be commutative.

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1 Introduction

Throughout the present paper all rings will be associative. A ring R equipped with an involution σ is said to be σ -prime if $aRb = aR\sigma(b) = 0$ implies that $a = 0$ or $b = 0$. Recall that a ring R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. Obviously, every prime ring with involution σ is σ -prime but the converse is in general not true. An ideal I of R is a σ -ideal if I is invariant under σ , i.e, $\sigma(I) = I$. In all that follows $Sa_\sigma(R)$ will denote the set of symmetric and skew symmetric elements of R , i.e, $Sa_\sigma(R) = \{x \in R/\sigma(x) = \pm x\}$. As usual the commutator $xy - yx$ will denoted by $[x, y] = xy - yx$. We shall use basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$. An additive mapping d of R to itself is a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$. A mapping $F : R \rightarrow R$ is said to be centralizing on a subset S of R if $[F(s), s] \in Z(R)$ for all s in S , where $Z(R)$ denotes the center of R ; in the special case where $[F(s), s] = 0$ for all s in S , the mapping F is said to be commuting on S .

The history of commuting and centralizing mappings goes back to 1955 when Divinsky [2] proved that a simple artinian ring is commutative if it has a

commuting non trivial automorphism. Two years later, Posner [6] have proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). This work was the motivation for our first result. More precisely, we are interested to know wether an analogous result can be proved for σ -prime rings with the restriction that the derivation is centralizing not on R but on a σ -ideal of R . The following theorem treats the case of σ -prime rings with characteristic not two.

Theorem 1 *Let R be a σ -prime ring with characteristic not two, I be a nonzero σ -ideal of R and d be a nonzero derivation of R which commutes with σ . If $[d(x), x] \in Z(R)$ for all x in I , then R is commutative.*

In the special case of a generalized inner derivation f of a ring R , i.e, $f(x) = ax + xb$ with $a, b \in R$, Brešar [1] remarked that the condition that f is centralizing on a subset S of R can be written in the form $d(x)x - xg(x) \in Z(R)$ for all $x \in S$, where d (resp. g) is the inner derivation induced by a (resp. b), that is $d(x) = [a, x]$ and $g(x) = [x, b]$. Motivating by this observation, Brešar introduced a more general concept than centralizing derivations by considering the situation when derivations f and g of a prime ring R satisfy $f(x)x - xg(x) \in Z(R)$ for all x in some subset S of R . He proved that a prime ring R , equipped with derivations $0 \neq f$ and g satisfying $f(x)x - xg(x) \in Z(R)$ for all x in a nonzero left ideal U of R , must be commutative. Our aim in the following theorem is to prove a similar result for σ -prime rings with characteristic not two.

Theorem 2 *Let R be a σ -prime ring with characteristic not two and I be a nonzero σ -ideal of R . Suppose there exist derivations d_1 and d_2 which commute with σ such that $d_1(x)x - xd_2(x) \in Z(R)$ for all $x \in I$. If $d_2 \neq 0$, then R is commutative.*

In [7], Kyoo-Hong Parck and Yong-Soo Sung state that if a prime ring R with characteristic different from two has a nonzero derivation d such that $[ad(x), x] = 0$ for all x in R , then $a = 0$ or R is commutative. The main purpose of the following theorem is to prove analogous result for σ -prime rings. The result we shall prove is:

Theorem 3 *Let R be a σ -prime ring with characteristic not two, I be a nonzero σ -ideal of R and $d \neq 0$ be a derivation of R which commutes with σ . If $[ad(x), x] = 0$ for all x in I , then $a = 0$ or R is commutative.*

P.H. Lee and T.L Lee [3] have shown that if a prime ring of characteristic different from two has a nonzero derivation d satisfying $[d(R), d(R)] \subset Z(R)$, then R is commutative. Our next goal is to generalize the result of [3] for σ -prime rings.

Theorem 4 *Let R be a σ -prime ring with characteristic not two, I be a nonzero σ -ideal of R and d a nonzero derivation such that $\sigma d = d\sigma$. If $d(I) \subset I$ and $[d(I), d(I)] \subset Z(R)$, then R is commutative.*

2 Proof of the main results

In all that follows, R will represent a σ -prime ring with characteristic not two. We will make frequent and important use of the following lemmas.

Lemma 1 (*[5], 3) of Theorem 1*) *Let I be a nonzero σ -ideal of R . If a, b in R are such that $aIb = 0 = aI\sigma(b)$, then $a = 0$ or $b = 0$.*

Proof. Suppose $a \neq 0$, there exists some $x \in I$ such that $ax \neq 0$. Indeed, otherwise

$$aRx = 0 \text{ and } aR\sigma(x) = 0 \text{ for all } x \in I$$

and therefore $a = 0$. Since $aIRb = 0$ and $aIR\sigma(b) = 0$, we then obtain

$$axRb = axR\sigma(b) = 0.$$

In view of the σ -primeness of R this yields $b = 0$. ■

Lemma 2 *Let I be a nonzero σ -ideal of R and $0 \neq d$ be a derivation on R which commutes with σ . If $[x, R]Id(x) = 0$ for all $x \in I$, then R is commutative.*

Proof. Let $x \in I$. Since $t = x - \sigma(x) \in I$, then $[t, r]Id(t) = 0$ for all $r \in R$. As $t \in Sa_\sigma(R)$, we then get

$$[t, r]Id(t) = \sigma([t, r])Id(t) = 0, \text{ for all } r \in R.$$

According to Lemma 1 we obtain

$$d(t) = 0 \text{ or } [r, t] = 0, \text{ for all } r \in R.$$

If $d(t) = 0$, then $d(x) = d(\sigma(x)) = \sigma(d(x))$. Therefore

$$0 = [x, r]Id(x) = [x, r]I\sigma(d(x))$$

and thus $d(x) = 0$ or $[r, x] = 0$ for all $r \in R$, by Lemma 1. Consequently, either $d(x) = 0$ or $x \in Z(R)$.

If $[r, t] = 0$ for all $r \in R$, then $t \in Z(R)$ and thus $[x, r] = [\sigma(x), r]$ for all $r \in R$. Hence

$$[x, r]Id(x) = 0 = \sigma([x, r])Id(x).$$

Again using Lemma 1, we get $d(x) = 0$ or $x \in Z(R)$.

In conclusion, for each $x \in I$ either $d(x) = 0$ or x in $Z(R)$. Let us consider

$G_1 = \{x \in I/d(x) = 0\}$ and $G_2 = \{x \in I / x \text{ in } Z(R)\}$. It is clear that G_1 and G_2 are additive subgroups of I such that $I = G_1 \cup G_2$. But a group can not be a union of two its proper subgroups and hence $I = G_1$ or $I = G_2$.

If $I = G_1$, then $d(x) = 0$ for all $x \in I$. For any $s \in R$, replace x by xs to get $xd(s) = 0$, for all $x \in I$ so that $Id(s) = 0$ for all $s \in R$. In particular

$$0 = 1Id(s) = \sigma(1)Id(s) \text{ for all } s \text{ in } R$$

and Lemma 1 gives $d = 0$, a contradiction. Hence, $I = G_2$ so that $I \subseteq Z(R)$. Let $r, s \in R$ and $x \in I$. From $rsx = rxs = srx$ we conclude that $[r, s]I = 0$ and then

$$[r, s]I1 = [r, s]I\sigma(1) = 0.$$

As $0 \neq 1$, in view of Lemma 1, we then get $[r, s] = 0$ for all $r, s \in R$. Therefore, R is a commutative ring. ■

Lemma 3 *Let I be a nonzero σ -ideal of R and $0 \neq d$ be a derivation on R which commutes with σ . If $[d(x), x] = 0$ for all $x \in I$, then R is commutative.*

Proof. Let $x, y \in I$; we linearize $[d(x), x] = 0$ to get

$$[d(x), y] + [d(y), x] = 0 \text{ for all } x, y \in I. \tag{1}$$

If in (1) we replace y by yx where $y \in I$ we obtain

$$[d(x), y]x + [d(y), x]x + [y, x]d(x) = 0 \text{ for all } x, y \in I.$$

Hence (1) yields

$$[x, y]d(x) = 0 \text{ for all } x, y \in I. \tag{2}$$

Thus, for any $r \in R$, we see that $0 = [x, ry]d(x) = [x, r]yd(x) = 0$ and therefore $[x, r]Id(x) = 0$ for all $r \in R, x \in I$. We then conclude, by Lemma 2, that R is commutative. ■

Lemma 4 *Let d be a derivation of R satisfying $d\sigma = \pm\sigma d$ and let I be a nonzero σ -ideal of R . If $d^2(I) = 0$ then $d = 0$.*

Proof. For any $x \in I$, we have $d^2(x) = 0$. Replacing x by xy , we obtain

$$d^2(x)y + 2d(x)d(y) + xd^2(y) = 0 \text{ for all } x, y \in I.$$

The fact that $d^2(I) = 0$ together with $\text{char}(R) \neq 2$, give $d(x)d(y) = 0$. If we replace x by xz where $z \in I$, in the last equality, we get $d(x)zd(y) = 0$ for all $x, y, z \in I$, so that $d(x)Id(y) = 0$. Since d commutes with σ we have, by Lemma 1, that

$$d(x) = 0 \text{ for all } x \in I. \tag{3}$$

In this replace x by xr where $r \in R$, to obtain $xd(r) = 0$ and therefore $IRd(r) = 0$ for all $r \in R$. Since I is a nonzero σ -ideal and R is σ -prime this last relation yields $d(r) = 0$ for all $r \in R$ and consequently $d = 0$. ■

Lemma 5 *Let d_1 and d_2 be derivations of R such that $d_1\sigma = \pm\sigma d_1$ and $d_2\sigma = \pm\sigma d_2$. If $I \neq 0$ is a σ -ideal of R such that $d_2(I) \subset I$ and $d_1 d_2(I) = 0$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. Let $u, v \in I$; then

$$0 = d_1 d_2(uv) = d_2(u)d_1(v) + d_1(u)d_2(v). \tag{4}$$

Replacing u by $d_2(u)$ in (4) we get

$$d_2^2(u)d_1(v) = 0 \text{ for all } u, v \in I. \tag{5}$$

If we replace v by vw where $w \in I$, in (5), we obtain

$$d_2^2(u)vd_1(w) = 0 \text{ for all } u, v, w \in I.$$

It then follows that $d_2^2(u)Id_1(w) = 0$ for all $u, w \in I$. The fact that $\sigma(I) = I$ combined with $d_1\sigma = \pm\sigma d_1$ give

$$d_2^2(u)Id_1(w) = d_2^2(u)I\sigma(d_1(w)) = 0$$

and we would conclude by Lemma 1 that either $d_1(w) = 0$ for all $w \in I$ so that $d_1 = 0$ or $d_2^2(u) = 0$ for all $u \in I$. If $d_2^2(u) = 0$ for all $u \in I$, we obtain from Lemma 4 that $d_2 = 0$. ■

Lemma 6 *Let d be a nonzero derivation of R which commutes with σ and I be a nonzero σ -ideal of R . If $a \in I \cap Sa_\sigma(R)$ is such that $[d(I), a] \subset Z(R)$, then $a \in Z(R)$.*

Proof. As $[d(a^2), a] = 2a[d(a), a]$, we have $a[d(a), a] \in Z(R)$ because $\text{char}(R) \neq 2$. This implies $[d(a), a]R[a, r] = 0$ for all $r \in R$. Since $a \in Sa_\sigma(R)$, then

$$[d(a), a]R[a, r] = [d(a), a]R\sigma([a, r]) = 0$$

and so $a \in Z(R)$ or $[d(a), a] = 0$. Let us suppose that $[d(a), a] = 0$. By our hypothesis we see that $[d[a, u], a] = [[d(a), u], a]$ which leads us to $[[d(a), u], a] \in Z(R)$ for all $u \in I$. If we replace u by au in the last relation, we obtain $a[[d(a), u], a] \in Z(R)$ for all $u \in I$. Hence, for $r \in R$, we have

$$ra[[d(a), u], a] = a[[d(a), u], a]r = ar[[d(a), u], a]$$

so that $[r, a]a[[d(a), u], a] = 0$. Let $t \in R$, from $[rt, a]a[[d(a), u], a] = 0$ we obtain $[r, a]t[[d(a), u], a] = 0$ and thus $[r, a]R[[d(a), u], a] = 0$. As $a \in Sa_\sigma(R)$, then

$$[r, a]R[[d(a), u], a] = \sigma([r, a])R[[d(a), u], a] = 0$$

and the σ -primeness of R yields $a \in Z(R)$ or $[[d(a), u], a] = 0$ for all $u \in I$. If $[[d(a), u], a] = 0$, then $[d(a), [u, a]] = 0$ for all $u \in I$, that is $d_{d(a)}d_a(u) = 0$ for all $u \in I$ where d_z is the inner derivation induced by z , i.e, $d_z(x) = [x, z]$. In view of Lemma 5, the fact that $d_{d(a)}(I) \subset I$ combined with $\sigma d_a = \pm d_a \sigma$ and $\sigma d_{d(a)} = \pm d_{d(a)}$ yield $d_a = 0$ or $d_{d(a)} = 0$ so that $a \in Z(R)$ or $d(a) \in Z(R)$. Let us now suppose that $d(a) \in Z(R)$; the fact that

$$[d(au), a] = d(a)[u, a] + a[d(u), a] \text{ for all } u \in I$$

forces $d(a)[u, a] + a[d(u), a] \in Z(R)$ for all $u \in I$. Then

$$0 = [d(a)[u, a] + a[d(u), a], a] = d(a)[[u, a], a] \text{ for all } u \in I$$

which gives $d(a)R[[u, a], a] = 0$ for all $u \in I$. Since $a \in Sa_\sigma(R)$ and d commutes with σ , then $d(a)R[[u, a], a] = \sigma(d(a))R[[u, a], a] = 0$ and the σ -primeness of R yields $d(a) = 0$ or $[[u, a], a] = 0$ for all $u \in I$. If $[[u, a], a] = 0$ for all $u \in I$, then the inner derivation d_a satisfies $d_a^2(I) = 0$. Since $d_a \sigma = \pm \sigma d_a$, according to Lemma 4, we conclude that $d_a = 0$ which proves $a \in Z(R)$. If $d(a) = 0$, then $a[d(u), a] = [d(au), a] \in Z(R)$ which would force $[a, r]R[d(u), a] = 0$ for all $u \in I, r \in R$. We then conclude $a \in Z(R)$ or $[d(u), a] = 0$ for all $u \in I$. Suppose that $[d(u), a] = 0$ for all $u \in I$, then

$$0 = [d(u[v, a]), a] = [d(u)[v, a], a] = d(u)[[v, a], a] \text{ for all } u, v \in I. \tag{6}$$

Replace u by uw where $w \in I$, in (6), to get

$$d(u)w[[v, a], a] = 0, \text{ for all } u, v, w \in I,$$

and whence

$$d(u)I[[v, a], a] = 0 = \sigma(d(u))I[[v, a], a] \text{ for all } u, v \in I.$$

As $d \neq 0$, according to Lemma 1, $[[v, a], a] = 0$ so that $d_a^2(v) = 0$ for all $v \in I$ and Lemma 4 yields $d_a = 0$; hence $a \in Z(R)$. ■

Proof of Theorem 1. Suppose that $[d(x), x] \in Z(R)$ for all $x \in I$, then

$$[d(x), y] + [d(y), x] \in Z(R) \text{ for all } x, y \in I.$$

In particular $[d(x), x^2] + [d(x)x + xd(x), x] \in Z(R)$ for all $x \in I$, so that

$$x[d(x), x] + [d(x), x]x + [d(x), x]x + x[d(x), x] \in Z(R) \text{ for all } x \in I.$$

Therefore $x[d(x), x] \in Z(R)$ for all $x \in I$. Thus, for any $r \in R$, we have that

$$rx[d(x), x] = x[d(x), x]r = xr[d(x), x]$$

and so

$$[r, x][d(x), x] = 0, \text{ for all } x \in I.$$

In particular, we obtain $[d(x), x]^2 = 0$ for all $x \in I$, so as $[d(x), x] \in Z(R)$ we then have

$$[d(x), x]R[d(x), x]\sigma([d(x), x]) \text{ for all } x \in I.$$

The fact that $[d(x), x]\sigma([d(x), x]) \in Sa_\sigma(R)$ combined with the σ -primeness of R yield

$$[d(x), x] = 0 \text{ or } [d(x), x]\sigma([d(x), x]) = 0.$$

Suppose that $[d(x), x]\sigma([d(x), x]) = 0$, then

$$[d(x), x]R\sigma([d(x), x]) = [d(x), x]R[d(x), x].$$

Again using the σ -primeness of R we get $[d(x), x] = 0$. In both cases we have

$$[d(x), x] = 0, \text{ for all } x \in I.$$

Applying Lemma 3, R is a commutative ring. ■

Proof of Theorem 2. If $I \cap Z(R) = 0$; as $d_1(x)x - xd_2(x) \in I \cap Z(R)$, then $d_1(x)x = xd_2(x)$ for all $x \in I$. Linearizing this equality we get

$$d_1(x)y + d_1(y)x = xd_2(y) + yd_2(x) \text{ for all } x, y \in I. \tag{7}$$

Replace y by yx in (7), the fact that $d_1(x)x = xd_2(x)$ combined with (7) assure that

$$[x, yd_2(x)] = 0 \text{ for all } x, y \in I. \tag{8}$$

In this replace y by ry , where $r \in R$, to get $[x, r]yd_2(x) = 0$ in such a way that

$$[x, R]Id_2(x) = 0 \text{ for all } x, z \in I.$$

As $d_2 \neq 0$, from Lemma 2 we conclude that R is commutative.

Now, assume that $I \cap Z(R) \neq 0$. Let $c \in I \cap Z(R)$; we can suppose that $\sigma(c) = \pm c$. Indeed, if $\sigma(c) \neq c$ we consider $t = c - \sigma(c)$, then $t \in I \cap Z(R)$ and $\sigma(t) = -t$. We linearize $d_1(x)x - xd_2(x) \in Z(R)$ to get

$$d_1(x)y + d_1(y)x - xd_2(y) - yd_2(x) \in Z(R) \text{ for all } x, y \in I. \tag{9}$$

Replace y by c in (9) and using $d_2(c) \in Z(R)$, to get

$$c(d_1(x) - d_2(x)) + (d_1(c) - d_2(c))x \in Z(R) \text{ for all } x \in I. \tag{10}$$

Taking $y = c^2$ in (9), in view of (10), we are forced to conclude that $c(d_1(c) - d_2(c))x \in Z(R)$. Hence, as I is a σ -ideal, we then obtain

$$c(d_1(c) - d_2(c))R[x, r] = 0 = c(d_1(c) - d_2(c))R\sigma([x, r]) \text{ for all } x \in I, r \in R$$

which leads us to $I \subset Z(R)$ and R is commutative by Lemma 3 or $d_1(c) = d_2(c)$. If $d_1(c) = d_2(c)$, then in view of (10) we get $c(d_1(x) - d_2(x)) \in Z(R)$ and so $d_1(x) - d_2(x) \in Z(R)$ for all $x \in I$. Therefore, $d(I) \subset Z(R)$ where $d = d_1 - d_2$. If $d \neq 0$, then it follows from Lemma 3 that R is commutative.

If $d_1 = d_2$ then

$$d_1(x)x - xd_1(x) = [d_1(x), x] \in Z(R) \text{ for all } x \in I.$$

and in view of Theorem 1 we are forced to conclude that R is commutative. ■

Proof of Theorem 3. Let x, y in I ; from $[ad(x+y), x+y] = 0$ it follows that

$$[ad(x), y] + [ad(y), x] = 0 \text{ for all } x, y \in I. \quad (11)$$

In this we replace y by yx , to obtain

$$[ad(x), yx] + [ad(y)x, x] + [ayd(x), x] = 0 \text{ for all } x, y \in I,$$

hence

$$[ad(x), y]x + [ad(y), x]x + ay[d(x), x] + [ay, x]d(x) = 0 \text{ for all } x, y \in I.$$

By (11) we conclude that

$$ay[d(x), x] + a[y, x]d(x) + [a, x]yd(x) = 0 \text{ for all } x, y \in I. \quad (12)$$

Replacing y by ay in (12), we get

$$a^2y[d(x), x] + a^2[y, x]d(x) + a[a, x]yd(x) + [a, x]ayd(x) = 0 \text{ for all } x, y \in I.$$

We have, using (12), that $[a, x]ayd(x) = 0$ for all $y \in I$ and therefore

$$[a, x]aId(x) = 0 \text{ for all } x \in I. \quad (13)$$

If $x \in I \cap Sa_\sigma(R)$, then $[a, x]aId(x) = [a, x]aI\sigma(d(x)) = 0$ and Lemma 1 gives $d(x) = 0$ or $[a, x]a = 0$. For any $y \in I$; as $y + \sigma(y) \in I \cap Sa_\sigma(R)$ we have that $d(y + \sigma(y)) = 0$ or $[a, y + \sigma(y)]a = 0$.

If $d(y + \sigma(y)) = 0$ then $d(y) = -\sigma(d(y))$ and so $d(y) = 0$ or $[a, y]a = 0$, by (13).

We now suppose $[a, y + \sigma(y)]a = 0$; similarly as $y - \sigma(y) \in I \cap Sa_\sigma(R)$ either $d(y - \sigma(y)) = 0$ or $[a, y - \sigma(y)]a = 0$. If $d(y - \sigma(y)) = 0$ then $d(y) = \sigma(d(y))$ and, using (13), we see that $d(y) = 0$ or $[a, y]a = 0$.

If $[a, y - \sigma(y)]a = 0$ then $0 = [a, y - \sigma(y)]a + [a, y + \sigma(y)]a = 2[a, y]a$, and since $\text{char}(R) \neq 2$ we have that $[a, y]a = 0$. Therefore, the additive group I is union of two subgroups A and B where $A = \{x \in I / d(x) = 0\}$ and $B = \{x \in I / [a, x]a = 0\}$. But an additive group can't be the union of two its proper subgroups. Thus $I = A$ or $I = B$. If $I = A$, that is $d(x) = 0$ for all

$x \in I$, then for any $r \in R$, $0 = d(rx) = d(r)x$. This implies that $d(r)I = 0$ and Lemma 1 gives $d = 0$, a contradiction. Consequently, $I = B$ and so $[a, x]a = 0$ for all $x \in I$. In this replace x by xy where $y \in I$, to get $[a, x]ya = 0$ so that $[a, x]Ia = 0 = [a, x]I\sigma(a)$. According to Lemma 1, we obtain $a = 0$ or $a \in Z(R)$. If $0 \neq a \in Z(R)$, we obtain $[ad(x), x] = a[d(x), x] = 0$ for all $x \in I$ and thus

$$aR[d(x), x] = 0 = \sigma(a)R[d(x), x] \text{ for all } x \in I.$$

But R is σ -prime so we are forced to $[d(x), x] = 0$ for all $x \in I$. Applying Lemma 3, R is then commutative. ■

Proof of Theorem 4. Let $I \neq 0$ be a σ -ideal of R with $[d(I), d(I)] \subset Z(R)$. Applying Lemma 6, this yields $d(I) \cap Sa_\sigma(R) \subset Z(R)$. Hence

$$d(u + \sigma(u)) \in Z(R) \text{ and } d(u - \sigma(u)) \in Z(R) \text{ for all } u \in I.$$

It follows that $2d(u) \in Z(R)$ for all $u \in I$. Thus $d(I) \subset Z(R)$ since $\text{char}(R) \neq 2$. In particular $[d(x), x] = 0$ for all $x \in I$ and using Lemma 3, we conclude that R is commutative. ■

Remark. As an immediate consequence of Theorem 4, if d is a nonzero derivation of R which commutes with σ and satisfying $0 \neq d^2(I) \subset Z(R)$, for a nonzero σ -ideal I of R , then R is commutative. Indeed, let $u, v \in I$; then $d^2([u, v]) = 2[d(u), d(v)] \in Z(R)$ and as $\text{char}(R) \neq 2$, we are forced to $[d(I), d(I)] \subset Z(R)$. Accordingly, R is commutative by Theorem 4.

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