On the Positive Solutions of the Difference Equation $x_{n+1} = \max \left\{ \frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_nx_{n-2}} \right\}$

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Abstract. We consider positive solutions of the following difference equation:

$$x_{n+1} = \max \left\{ \frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_nx_{n-2}} \right\}$$

where $A, B$ are any positive coefficients and the initial values $x_0, x_{-1}, x_{-2}$ are any positive numbers.

Keywords: Difference Equation, Periodic solution, Equilibrium point

1. INTRODUCTION

In this paper we investigate the periodic character of the positive solutions of the difference equation

$$(1.1) \quad x_{n+1} = \max \left\{ \frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_nx_{n-2}} \right\}$$

where $A, B$ are any positive coefficients and the initial values $x_0, x_{-1}, x_{-2}$ are any positive numbers.

For difference equations with maximum which are related to the Lynees equation, see [1-6]. We investigate Eq. (1.1) similar to [2].

2. PRELIMINARIES

Let $A, B, x_0, x_1, x_2$ be positive numbers. The equation
(2.1) \[ x_{n+1} = \max \left\{ \frac{A}{x_n^2}, \frac{B x_{n-1}}{x_n x_{n-2}} \right\} \quad \text{for } n \geq 2, \]

defines a sequence of positive numbers \( x_n, \ n = 0, 1, 2, \ldots \). It is convenient to define a sequence \( \{y_n\} \) as follows:

(2.2) \[ y_{n+1} = \frac{x_{n+1} x_n^2}{B} \quad \text{for } n = 0, 1, 2, \ldots. \]

**Lemma 1.** The sequence \( \{y_n\} \) defined by (2.2) with positive initial conditions \( y_0 \) and \( y_1 \) satisfies the equation

(2.3) \[ y_{n+1} = \max \left\{ C, \frac{y_n}{y_{n-1}} \right\} \]

where \( C = A B^{-1} > 0. \)

*Proof.* It is sufficient to notice that equation (2.1) is equivalent to

\[ \frac{x_{n+1} x_n^2}{B} = \max \left\{ \frac{A}{B}, \frac{x_n x_{n-1}}{x_{n-2}} \right\} \]

since \( B \) and \( x_n \) are positive. \( \blacksquare \)

We will now study the positive solutions of equation (2.3).

**Lemma 2.** The sequence \( \{y_n\} \) generated by equation (2.3) \( y_0, y_1 > 0 \) satisfies the inequalities

(2.4) \[ C \leq y_n \leq \max \{C, 1/C\} \quad \text{for } n \geq 5. \]

*Proof.* Equation (2.3) immediately implies

(2.5) \[ C \leq y_n \quad \text{for } n \geq 2 \]

Assume that for some \( k \geq 5 \), we have

(2.6) \[ y_k > \max \{C, 1/C\}. \]

Then it follows from (2.3) that \( y_k = \frac{y_{k-1}}{y_{k-2}}. \) By using (2.3) again, we obtain
\[ y_k y_{k-2} = y_{k-1} = \max \left\{ C, \frac{y_{k-2}}{y_{k-3}} \right\}. \]

Dividing the first and last expressions by \( y_{k-2} \) yields

\[(2.7) \quad y_k = \max \left\{ \frac{C}{y_{k-2}}, \frac{1}{y_{k-3}} \right\}. \]

In view of (2.5), we have \( \frac{C}{y_{k-2}} \leq 1 \) and \( \frac{1}{y_{k-3}} \leq \frac{1}{C} \) where \( k - 3 \geq 2 \). Hence

\[ \max \left\{ \frac{C}{y_{k-2}}, \frac{1}{y_{k-3}} \right\} \leq \max \{1, 1/C\} \leq \max \{C, 1/C\}. \]

From (2.7), we conclude that

\[ y_k \leq \max \{C, 1/C\} \]

which contradicts (2.6). This completes the proof.

**Lemma 3.** If \( C < 1 \), then

\[(2.8) \quad y_n y_{n+2} = y_{n+1} \text{ for } n \geq 6. \]

**Proof.** Since \( y_{n+1} > 0 \) equation (2.3) implies that

\[(2.9) \quad \frac{y_{n+2}}{y_{n+1}} = \max \left\{ \frac{C}{y_{n+1}}, \frac{1}{y_n} \right\}. \]

From Lemma 2.2, since \( C < 1 \), it follows that \( \frac{1}{y_n} \geq C \), for \( n \geq 5 \). Hence, in view of (2.9), we have \( \frac{y_{n+2}}{y_{n+1}} \geq C \). Then equation (2.3) implies

\[ y_{n+3} = \max \left\{ C, \frac{y_{n+2}}{y_{n+1}} \right\} = \frac{y_{n+2}}{y_{n+1}} \]

which completes the proof.

**Lemma 4.** If \( C \geq 1 \), then \( y_n = C \), for \( n \geq 5 \).

The proof follows immediately from Lemma 2.
3. MAIN RESULTS

We assume first that \( A \geq B \) and later that \( A < B \). It turns out the case \( A \geq B \) is simpler and is handled by following theorem.

**Theorem 5.** Let \( A \geq B > 0 \) and \( x_0, x_1, x_2 > 0 \). Then all solutions of equation (2.1) are either

\[
x_{2n} = \frac{A^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}}}{A^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}} + B^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}}}, \quad x_{2n+1} = \frac{(x_5)^{2^{n-4}}}{A^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}} + B^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}}}
\]

or eventually constant (for \( n \geq 3 \)).

**Proof.** Note that \( C \geq 1 \) since \( A \geq B \). In view of the change of variables (2.2) and Lemma 4, we have

\[
x_n x_{n-1} = y_n B = CB = A \text{ for } n \geq 5.
\]

A simple induction argument yields that

\[
x_{2n} = \left(\frac{x_5}{A^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}} + B^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}} + C^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}}}\right)^{2^{n-3}}, \quad x_{2n+1} = \frac{(x_5)^{2^{n-4}}}{A^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}} + B^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}} + C^{\frac{1}{n+3} \sum_{k=1}^{2^{n-3}}}}
\]

for \( n \geq 3 \).

If \( x_5 = A = 1 \), then \( x_n = 1 \) for \( n \geq 5 \).

**Theorem 6.** Let \( C < 1 \). Then every solution \( y_n \) of equation (2.3) is either a period six solution or \( y_n = 1 \) for all \( n \geq 0 \).

**Proof.** We first observe from Lemma 3 that

\[
y_n y_{n+3} = y_n y_{n+2} y_{n+1} y_{n+2} = y_{n+1} \frac{1}{y_{n+1}} = 1 \text{ for } n \geq 6.
\]

Then (3.1) implies that

\[
y_n = y_{n+6} \text{ for } n \geq 6.
\]

Therefore, \( y_n \) is an eventually periodic sequence. Let \( m \) be its period, i.e., the smallest positive integer \( m \) for which

\[
y_{n+m} = y_n \text{ for } n \geq 6.
\]

It is clear that \( m \leq 6 \) and \( m \) is a divisor of 6, in view of (3.2). Thus \( m \in \{1, 2, 3, 6\} \). Next we prove that \( m \) is neither 2 nor 3. If \( m = 3 \), then (3.3), (3.1), and the positivity of \( y_n \) imply \( y_n = 1 \), for all \( m \geq 6 \). This contradicts the definition of \( m \). If \( m = 2 \), equations (3.3) and (3.1) yield

\[
y_n y_{n+1} = 1 \text{ for } n \geq 6.
\]
Then (3.4), Lemma 3, and the fact that $m = 2$ imply that

$$y_n = y_{n+1} = y_n y_n y_{n+1} = y_n y_n y_{n+2} = y_n y_n y_n y_n \text{ or } y_n = 1 \text{ for } n \geq 6.$$  

This again contradicts the definition of $m$. Thus, $m$ is either 1 or 6. Let us first consider the case $m = 1$. Equation (3.3) implies that $y_n = y_0$ for $n \geq 6$, which together with Lemma 2.3 yields $y_n = 1$, for $n \geq 6$. Now, by substituting $n = 6$ in (2.3), we have

$$1 = y_7 = \max \left\{ C, \frac{y_6}{y_5} \right\}$$

and, therefore, $y_5 = y_6$ because $C < 1$. Then $y_n = 1$ for $n \geq 5$. Continuing in this fashion, we can prove that $y_n = 1$ for $n \geq 0$. In fact, $m = 1$ is equivalent to $y_0 = y_1 = 1$. Therefore $m = 6$, provided that $y_0$ and $y_1$ are not both equal to 1.

This proof is completed.

**Theorem 7.** Let $0 < A < B$ and $x_n$ be the solutions of equation (2.1) generated by a given set of positive numbers $\{x_0, x_1, x_2\}$. If $x_0 = x_1 = x_2 = \sqrt{B}$, then the solutions of equation (2.1) are constant.

**Proof.** From Lemma 3 and equation (2.2), it follows that

$$\frac{x_{n+2} x_n^2}{x_n} = B \text{ for } n \geq 6.$$  

Then we have $x_n = x_n \frac{x_n B}{x_{n+2} x_n} = \frac{x_n B}{x_{n+2} x_n}$, then

$$x_{n+2} = \frac{x_n B}{x_{n+1}}.$$  

(3.5)

Since $x_0 = x_1 = x_2 = \sqrt{B}$, $x_n = \sqrt{B}$ for $n \geq 3$.  

**Theorem 8.** If $A \geq B$ then equilibrium point of equation (1.1) is $\overline{x} = \sqrt[3]{A}$. Also if $A < B$, then equilibrium point of equation (1.1) is $\overline{x} = \sqrt[3]{B}$.

**Proof.** If $A \geq B$, then $\overline{x} = \max \left\{ \frac{A}{\overline{x}^2}, \frac{B}{\overline{x}^2} \right\}$. Thus,

$$\overline{x} = \frac{A}{\overline{x}^2} \text{ and } \overline{x}^3 = A \text{ and } \overline{x} = \sqrt[3]{A}.$$  

If $A < B$, then $\overline{x} = \max \left\{ \frac{A}{\overline{x}^2}, \frac{B}{\overline{x}^2} \right\}$. Thus,

$$\overline{x} = \frac{B}{\overline{x}^2} \text{ and } \overline{x}^3 = B \text{ and } \overline{x} = \sqrt[3]{B}.$$  

\[\square\]
Corollary 9. Equilibrium point of equation (1.1) is \( \bar{x} = \max \left\{ \sqrt[3]{A}, \sqrt[3]{B} \right\} \).

The proof follows immediately from Theorem 3.4.

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