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On the Positive Solutions of the Difference Equation $\mathbf{x}_{n+1} = \max\left\{\frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_n x_{n-2}^2}\right\}$

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Abstract. We consider positive solutions of the following difference equation:

$$x_{n+1} = \max\left\{\frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_n x_{n-2}^2}\right\}$$

where *A, B* are any positive coefficients and the initial values x_0, x_{-1}, x_{-2} are any positive numbers.

Keywords: Difference Equation, Periodic solution, Equiblirium point

1. INTRODUCTION

In this paper we investigate the periodic character of the positive solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_n x_{n-2}^2}\right\} \tag{1.1}$$

where *A, B* are any positive coefficients and the initial values x_0, x_{-1}, x_{-2} are any positive numbers.

For difference equations with maximum which are related to the Lynees equation, see [1-6]. We investigate Eq. (1.1) similar to [2].

2. PRELIMINARIES

Let *A, B,* x_0, x_1, x_2 be positive numbers. The equation

$$x_{n+1} = \max\left\{\frac{A}{x_n^2}, \frac{Bx_{n-1}}{x_n x_{n-2}^2}\right\} \text{ for } n \geq 2, \tag{2.1}$$

defines a sequence of positive numbers $x_n,\ n = 0, 1, 2, ...$ It is convenient to define a sequence $\{y_n\}$ as follows :

$$y_{n+1} = \frac{x_{n+1}x_n^2}{B} \text{ for } n = 0, 1, 2, ... \tag{2.2}$$

Lemma 1. *The sequence $\{y_n\}$ defined by (2.2) with positive initial conditions y_0 and y_1 satisfies the equation*

$$y_{n+1} = \max\left\{C, \frac{y_n}{y_{n-1}}\right\} \tag{2.3}$$

where $C = A.B^{-1} > 0.$

Proof. It is sufficient to notice that equation (2.1) is equivalent to

$$\frac{x_{n+1}x_n^2}{B} = \max\left\{\frac{A}{B}, \frac{x_n x_{n-1}}{x_{n-2}^2}\right\}$$

since B and x_n are positive. ∎

We will now study the positive solutions of equation (2.3).

Lemma 2. *The sequence $\{y_n\}$ generated by equation (2.3) y_0, $y_1 > 0$ satisfies the inequalities*

$$C \leq y_n \leq \max\{C, 1/C\} \text{ for } n \geq 5. \tag{2.4}$$

Proof. Equation (2.3) immediately implies

$$C \leq y_n \text{ for } n \geq 2 \tag{2.5}$$

Assume that for some $k \geq 5$, we have

$$y_k > \max\{C, 1/C\}. \tag{2.6}$$

Then it follows from (2.3) that $y_k = \frac{y_{k-1}}{y_{k-2}}$. By using (2.3) again, we obtain

$$y_k y_{k-2} = y_{k-1} = \max\left\{C, \frac{y_{k-2}}{y_{k-3}}\right\}.$$

Dividing the first and last expressions by y_{k-2} yields

$$y_k = \max\left\{\frac{C}{y_{k-2}}, \frac{1}{y_{k-3}}\right\}. \tag{2.7}$$

In view of (2.5), we have $\frac{C}{y_{k-2}} \leq 1$ and $\frac{1}{y_{k-3}} \leq \frac{1}{C}$ where $k-3 \geq 2$. Hence

$$\max\left\{\frac{C}{y_{k-2}}, \frac{1}{y_{k-3}}\right\} \leq \max\{1, 1/C\} \leq \max\{C, 1/C\}.$$

From (2.7), we conclude that

$$y_k \leq \max\{C, 1/C\}$$

which contradicts (2.6). This completes the proof. ∎

Lemma 3. *If* $C < 1$, *then*

$$y_n y_{n+2} = y_{n+1} \text{ for } n \geq 6. \tag{2.8}$$

Proof. Since $y_{n+1} > 0$ equation (2.3) implies that

$$\frac{y_{n+2}}{y_{n+1}} = \max\left\{\frac{C}{y_{n+1}}, \frac{1}{y_n}\right\}. \tag{2.9}$$

From Lemma 2.2, since $C < 1$, it follows that $\frac{1}{y_n} \geq C$, for $n \geq 5$. Hence, in view of (2.9), we have $\frac{y_{n+2}}{y_{n+1}} \geq C$. Then equation (2.3) implies

$$y_{n+3} = \max\left\{C, \frac{y_{n+2}}{y_{n+1}}\right\} = \frac{y_{n+2}}{y_{n+1}}$$

which completes the proof. ∎

Lemma 4. *If* $C \geq 1$, *then* $y_n = C$, *for* $n \geq 5$.

The proof follows immediately from Lemma 2.

3. MAIN RESULTS

We assume first that $A \geq B$ and later that $A < B$. It turns out the case $A \geq B$ is simplier and is handled by following theorem.

Theorem 5. *Let $A \geq B > 0$ and $x_0,\ x_1,\ x_2 > 0$. Then all solutions of equation (2.1) are either* $x_{2n} = \frac{A^{1+\sum_{k=1}^{n-3} 2^{2k-1}}}{(x_5)^{2^{2n-5}}}$, $x_{2n+1} = \frac{(x_5)^{2^{2n-4}}}{A^{1+\sum_{k=1}^{n-3} 2^{2k}}}$ *or eventually constant (for $n \geq 3$).*

Proof. Note that $C \geq 1$ since $A \geq B$. In view of the change of variables (2.2) and Lemma 4, we have

$$x_n x_{n-1}^2 = y_n B = CB = A \ \text{ for } n \ \geq 5.$$

A simple induction argument yields that

$$x_{2n} = \frac{A^{1+\sum_{k=1}^{n-3} 2^{2k-1}}}{(x_5)^{2^{2n-5}}},\ x_{2n+1} = \frac{(x_5)^{2^{2n-4}}}{A^{1+\sum_{k=1}^{n-3} 2^{2k}}} \text{ for } n \geq 3.$$

If $x_5 = A = 1$, then $x_n = 1$ for $n \geq 5$. ∎

Theorem 6. *Let $C < 1$. Then every solution y_n of equation (2.3) is either a period six solution or $y_n = 1$ for all $n \geq 0$.*

Proof. We first observe from Lemma 3 that

$$y_n y_{n+3} = y_n y_{n+2} \frac{y_{n+3}}{y_{n+2}} = y_{n+1} \frac{1}{y_{n+1}} = 1 \text{ for } n \geq 6. \tag{3.1}$$

Then (3.1) implies that

$$y_n = y_{n+6} \text{ for } n \geq 6. \tag{3.2}$$

Therefore, y_n is an eventually periodic sequence. Let m be its period, i.e., the smallest positive integer m for which

$$y_{n+m} = y_n \text{ for } n \geq 6. \tag{3.3}$$

It is clear that $m \leq 6$ and m is a divisior of 6, in view of (3.2). Thus $m \in \{1, 2, 3, 6\}$. Next we prove that m is neither 2 nor 3. If $m = 3$, then (3.3), (3.1), and the positivity of y_n imply $y_n = 1$, for all $m \geq 6$. This contradicts the definition of m. If $m = 2$, equations (3.3) and (3.1) yield

$$y_n y_{n+1} = 1 \text{ for } n \geq 6. \tag{3.4}$$

Then (3.4), Lemma 3, and the fact that $m = 2$ imply that

$$y_n = y_n 1 = y_n y_n y_{n+1} = y_n y_n y_n y_{n+2} = y_n y_n y_n y_n \text{ } or \text{ } y_n = 1 \text{ for } n \geq 6.$$

This again contradicts the definition of m.Thus, m is either 1 or 6. Let us first consider the case $m = 1$. Equation (3.3) implies that $y_n = y_6$ for $n \geq 6$, which together with Lemma 2.3 yields $y_n = 1$, for $n \geq 6$.Now, by substituting $n = 6$ in (2.3), we have

$$1 = y_7 = \max\left\{C, \frac{y_6}{y_5}\right\}$$

and, therefore, $y_5 = y_6$ because $C < 1$. Then $y_n = 1$ for $n \geq 5$. Continuing in this fashion, we can prove that $y_n = 1$ for $n \geq 0$. In fact, $m = 1$ is equivalent to $y_0 = y_1 = 1$. Therefore $m = 6$, provided that y_0 and y_1 are not both equal to 1.

This proof is completed. ∎

Theorem 7. *Let $0 < A < B$ and x_n be the solutions of equation (2.1) generated by a given set of positive numbers $\{x_0, x_1, x_2\}$. If $x_0 = x_1 = x_2 = \sqrt{B}$, then the solutions of equation (2.1) are constant.*

Proof. From Lemma 3 and equation (2.2), it follows that

$$\frac{x_{n+2}x_{n-1}^2}{x_n} = B \text{ for } n \geq 6.$$

Then we have $x_n = x_n \frac{x_n B}{x_{n+2}x_{n-1}^2}$, then

$$x_{n+2} = \frac{x_n B}{x_{n-1}^2}. \tag{3.5}$$

Since $x_0 = x_1 = x_2 = \sqrt{B}$, $x_n = \sqrt{B}$ for $n \geq 3$. ∎

Theorem 8. *If $A \geq B$ then equilibrium point of equation(1.1) is $\overline{x} = \sqrt[3]{A}$. Also If $A < B$, then equilibrium point of equation(1.1) is $\overline{x} = \sqrt[3]{B}$.*

Proof. If $A \geq B$, then $\overline{x} = \max\left\{\frac{A}{\overline{x}^2}, \frac{B}{\overline{x}^2}\right\}$. Thus,

$$\overline{x} = \frac{A}{\overline{x}^2} \textit{and } \overline{x}^3 = A \text{ and } \overline{x} = \sqrt[3]{A}.$$

If $A < B$, then $\overline{x} = \max\left\{\frac{A}{\overline{x}^2}, \frac{B}{\overline{x}^2}\right\}$. Thus,

$$\overline{x} = \frac{B}{\overline{x}^2} \textit{and } \overline{x}^3 = B \text{ and } \overline{x} = \sqrt[3]{B}.$$

∎

Corollary 9. *Equilibrium point of equation(1.1) is* $\overline{x} = \max\left\{\sqrt[3]{A}, \sqrt[3]{B}\right\}$.

The proof follows immediately from Theorem 3.4.

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