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On the Solutions of the Difference Equation $x_{n+1} = \max\left\{\frac{1}{x_{n-1}}, x_{n-1}\right\}$

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Abstract. We study the behaviour of the solutions of the following difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_{n-1}}, x_{n-1}\right\},$$

where $x_{-1}$and x_0 are nonzero real numbers. In most of the cases we determine the behaviour of the solutions in function initial conditions $x_{-1}$and x_0.

1. INTRODUCTION

In this paper we study the behaviour of the solutions of the following difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_{n-1}}, x_{n-1}\right\} \tag{1}$$

where $x_{-1}$and x_0 are nonzero real numbers.

Some closely related equations were investigated, for example [1-5]. For example, the investigation of the difference equation

(2) $$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, ..., \frac{A_k}{x_{n-k}}\right\}, n = 0, 1, ...$$

where A_i, $i = 0, 1, ..., k$, are real numbers, such that at least one of them is different from zero and initial conditions $x_0, x_{-1}, ..., x_{-k}$, are different from zero, was proposed in [3] and [4].

A special case the max operator in Eq.(2) arises naturally in certain models in automatic control theory (see, [6,7]).

2. MAIN RESULT

Theorem 1. *Consider the difference equation (1) for* $0 < x_0, x_{-1}$.

a) *If* $0 < x_{-1}, x_0 < 1$ *then*

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...).$$

b)*If* $0 < x_{-1} < 1$ *and* $x_0 > 1$ *then*

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-1}}, x_0, ...).$$

c)*If* $x_{-1} > 1$ *and* $0 < x_0 < 1$ *then*

$$(x_n) = (x_{-1}, x_0, x_{-1}, \frac{1}{x_0}, x_{-1}, \frac{1}{x_0}, ...).$$

d) *If* $x_{-1}, x_0 > 1$ *then*

$$(x_n) = (x_{-1}, x_0, x_{-1}, x_0, x_{-1}, x_0, ...).$$

Proof. a) Let $0 < x_0, x_{-1}$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, 0 < x_{-1} < 1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, 0 < x_0 < 1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, 0 < x_{-1} < 1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, 0 < x_0 < 1$.

By induction we obtain that $x_{2n} = \frac{1}{x_{-1}}$ and $x_{2n-1} = \frac{1}{x_0}$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...)$.

b) Let $0 < x_0, x_{-1}$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, 0 < x_{-1} < 1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, x_0 > 1$,then $x_3 = \max\left\{x_{-1}, \frac{1}{x_{-1}}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, 0 < x_{-1} < 1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, x_0 > 1$.

By induction we obtain that $x_{2n} = \frac{1}{x_{-1}}$ and $x_{2n-1} = x_0$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-1}}, x_0, ...)$.

c) Let $0 < x_0, x_{-1}$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, x_{-1} > 1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, 0 < x_0 < 1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, x_{-1} > 1$, then $x_4 = \max\left\{x_0, \frac{1}{x_0}\right\} = \frac{1}{x_0}, 1 > x_0^2, 0 < x_0 < 1$.

By induction we obtain that $x_{2n} = x_{-1}$ and $x_{2n-1} = \frac{1}{x_0}$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, x_{-1}, \frac{1}{x_0}, x_{-1}, \frac{1}{x_0}, ...)$.

d) Let $0 < x_0, x_{-1}$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, x_{-1} > 1$, then$x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, x_0 > 1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, x_{-1} > 1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, x_0 > 1$.

By induction we obtain that $x_{2n} = x_{-1}$ and $x_{2n-1} = x_0$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, x_{-1}, x_0, x_{-1}, x_0, ...)$. ∎

Theorem 2. *Consider the difference equation (1) for $x_0 < 0 < x_{-1}$.*

a) *If $0 < x_{-1} < 1$ and $x_0 < -1$ then*

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...).$$

b)*If $0 < x_{-1} < 1$ and $-1 < x_0 < 0$ then*

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-1}}, x_0, ...).$$

c)*If $x_{-1} > 1$ and $x_0 < -1$then*

$$(x_n) = (x_{-1}, x_0, x_{-1}, \frac{1}{x_0}, x_{-1}, \frac{1}{x_0}, ...).$$

d) *If $x_{-1} > 1$ and $-1 < x_0 < 0$then*

$$(x_n) = (x_{-1}, x_0, x_{-1}, x_0, x_{-1}, x_0, ...).$$

Proof. a) Let $x_0 < 0 < x_{-1}$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, 0 < x_{-1} < 1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, x_0 < -1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, 0 < x_{-1} < 1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, x_0 < -1$.

By induction we obtain that $x_{2n} = \frac{1}{x_{-1}}$ and $x_{2n-1} = \frac{1}{x_0}$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...)$.

b) Let $x_0 < 0 < x_{-1}$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, 0 < x_{-1} < 1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, -1 < x_0 < 0$,then $x_3 = \max\left\{x_{-1}, \frac{1}{x_{-1}}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, 0 < x_{-1} < 1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, -1 < x_0 < 0$.

By induction we obtain that $x_{2n} = \frac{1}{x_{-1}}$ and $x_{2n-1} = x_0$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-1}}, x_0, ...)$.

c) Let $x_0 < 0 < x_{-1}$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, x_{-1} > 1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, x_0 < -1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, x_{-1} > 1$, then $x_4 = \max\left\{x_0, \frac{1}{x_0}\right\} = \frac{1}{x_0}, 1 > x_0^2, x_0 < -1$.

By induction we obtain that $x_{2n} = x_{-1}$ and $x_{2n-1} = \frac{1}{x_0}$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, x_{-1}, \frac{1}{x_0}, x_{-1}, \frac{1}{x_0}, ...)$.

d) Let $x_0 < 0 < x_{-1}$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, x_{-1} > 1$, then$x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, -1 < x_0 < 0$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, x_{-1} > 1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, -1 < x_0 < 0$.

By induction we obtain that $x_{2n} = x_{-1}$ and $x_{2n-1} = x_0$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, x_{-1}, x_0, x_{-1}, x_0, ...)$ ∎

Theorem 3. *Consider the equation (1),*

a) *If* $x_{-1} < -1$ *and* $0 < x_0 < 1$ *then*

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...).$$

b) *If* $x_{-1} < -1$ *and* $x_0 > 1$ *then*

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-1}}, x_0, ...).$$

c) *If* $-1 < x_{-1} < 0$ *and* $0 < x_0 < 1$ *then*

$$(x_n) = (x_{-1}, x_0, x_{-1}, \frac{1}{x_0}, x_{-1}, \frac{1}{x_0}, ...).$$

d) *If* $-1 < x_{-1} < 0$ *and* $x_0 > 1$ *then*

$$(x_n) = (x_{-1}, x_0, x_{-1}, x_0, x_{-1}, x_0, ...).$$

Proof. a) Let $x_{-1} < 0 < x_0$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, x_{-1} < -1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, 0 < x_0 < 1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, x_{-1} < -1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, 0 < x_0 < 1$.

By induction we obtain that $x_{2n} = \frac{1}{x_{-1}}$ and $x_{2n-1} = \frac{1}{x_0}$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...)$.

b) Let $x_{-1} < 0 < x_0$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, x_{-1} < -1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, x_0 > 1$,then $x_3 = \max\left\{x_{-1}, \frac{1}{x_{-1}}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, x_{-1} < -1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, x_0 > 1$.

By induction we obtain that $x_{2n} = \frac{1}{x_{-1}}$ and $x_{2n-1} = x_0$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-1}}, x_0, ...)$.

c) Let $x_{-1} < 0 < x_0$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, -1 < x_{-1} < 0$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, 0 < x_0 < 1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, -1 < x_{-1} < 0$, then $x_4 = \max\left\{x_0, \frac{1}{x_0}\right\} = \frac{1}{x_0}, 1 > x_0^2, 0 < x_0 < 1$.

By induction we obtain that $x_{2n} = x_{-1}$ and $x_{2n-1} = \frac{1}{x_0}$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, x_{-1}, \frac{1}{x_0}, x_{-1}, \frac{1}{x_0}, ...)$.

d) $x_{-1} < 0 < x_0$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, -1 < x_{-1} < 0$, then$x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, x_0 > 1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, -1 < x_{-1} < 0$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, x_0 > 1$.

By induction we obtain that $x_{2n} = x_{-1}$ and $x_{2n-1} = x_0$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, x_{-1}, x_0, x_{-1}, x_0, ...)$. ∎

Theorem 4. *Consider the difference equation (1) for* $x_{-1}, x_0 < 0$.

a) *If* $x_{-1}, x_0 < -1$ *then*

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...).$$

b) *If* $x_{-1} < -1$ *and* $-1 < x_0 < 0$ *then*

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-1}}, x_0, ...).$$

c) *If* $-1 < x_{-1} < 0$ *and* $x_0 < -1$ *then*

$$(x_n) = (x_{-1}, x_0, x_{-1}, \frac{1}{x_0}, x_{-1}, \frac{1}{x_0}, ...).$$

d) *If* $-1 < x_{-1}, x_0 < 0$ *then*

$$(x_n) = (x_{-1}, x_0, x_{-1}, x_0, x_{-1}, x_0, ...).$$

Proof. a) Let $x_{-1}, x_0 < 0$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, x_{-1} < -1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, x_0 < -1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, x_{-1} < -1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, x_0 < -1$.

By induction we obtain that $x_{2n} = \frac{1}{x_{-1}}$ and $x_{2n-1} = \frac{1}{x_0}$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{1}{x_{-1}}, \frac{1}{x_0}, ...)$.

b) Let $x_{-1}, x_0 < 0$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, x_{-1} < -1$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, -1 < x_0 < 0$,then $x_3 = \max\left\{x_{-1}, \frac{1}{x_{-1}}\right\} = \frac{1}{x_{-1}}, 1 > x_{-1}^2, x_{-1} < -1$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, -1 < x_0 < 0$.

By induction we obtain that $x_{2n} = \frac{1}{x_{-1}}$ and $x_{2n-1} = x_0$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, \frac{1}{x_{-1}}, x_0, \frac{1}{x_{-1}}, x_0, ...)$.

c) Let$x_{-1}, x_0 < 0$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, -1 < x_{-1} < 0$, then $x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = \frac{1}{x_0}, 1 > x_0^2, x_0 < -1$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, -1 < x_{-1} < 0$, then $x_4 = \max\left\{x_0, \frac{1}{x_0}\right\} = \frac{1}{x_0}, 1 > x_0^2, x_0 < -1$.

By induction we obtain that $x_{2n} = x_{-1}$ and $x_{2n-1} = \frac{1}{x_0}$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, x_{-1}, \frac{1}{x_0}, x_{-1}, \frac{1}{x_0}, ...)$.

d) $x_{-1}, x_0 < 0$, then $x_1 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, -1 < x_{-1} < 0$, then$x_2 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, -1 < x_0 < 0$,then $x_3 = \max\left\{\frac{1}{x_{-1}}, x_{-1}\right\} = x_{-1}, x_{-1}^2 > 1, -1 < x_{-1} < 0$, then $x_4 = \max\left\{\frac{1}{x_0}, x_0\right\} = x_0, x_0^2 > 1, -1 < x_0 < 0$.

By induction we obtain that $x_{2n} = x_{-1}$ and $x_{2n-1} = x_0$ for $n \geq 0$, that is $(x_n) = (x_{-1}, x_0, x_{-1}, x_0, x_{-1}, x_0, ...)$.

Which completes the proof of theorem. ∎

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