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On the Recursive Sequence

$x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$

Dağıstan Simsek

Selcuk University, Engineering-Architecture Faculty
Department of Industrial Engineering
42075 Kampüs/Konya, Turkey
dsimsek@selcuk.edu.tr

Cengiz Cinar and Ibrahim Yalcinkaya

Mathematics Department, Faculty of Education
Selcuk University, 42090, Konya, Turkey
ccinar@selcuk.edu.tr, iyalcinkaya@selcuk.edu.tr

Abstract. In this paper a solution of the following difference equation was investigated

$$x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}, \ n = 0, 1, 2, ...$$

where $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

1. INTRODUCTION

Recently there has been a lot of interest in studying the periodic nature of nonlinear difference equations. For some recent result concerning among other problems, the periodic nature of scalar nonlinear difference equations see, for examples [1,2,4,5].

In [3] the following problem was posed.Is there a solution of the following difference equation

$$x_{n+1} = \frac{\beta x_{n-1}}{\beta + x_n} \text{ for } n = 0, 1, 2, ...$$

where $x_{-1}, x_0, \beta \in (0, \infty)$ such that $x_n \to 0$ as $n \to \infty$.

In [6] Stevic assumed that $\beta = 1$ and solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1+x_n} \text{ for } n = 0, 1, 2, ...$$

where $x_{-1}, x_0 \in (0, \infty)$. Also, this result was generalized to the equation of the following form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)} \text{ for } n = 0, 1, 2, ...$$

where $x_{-1}, x_0 \in (0, \infty)$.

In this paper we investigated the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}} \text{ for } n = 0, 1, 2, ... \tag{1}$$

where $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

2. MAIN RESULT

Theorem 1. *Consider the difference equation (1). Then the following statements are true.*

a) *The sequences* $(x_{4n-3}), (x_{4n-2}), (x_{4n-1})$ *and* (x_{4n}) *are decreasing and there exist* $p, q, r, s \geq 0$ *such that*

$$\lim_{n\to\infty} x_{4n-3} = p, \ \lim_{n\to\infty} x_{4n-2} = q, \ \lim_{n\to\infty} x_{4n-1} = r \ \text{ and } \lim_{n\to\infty} x_{4n} = s.$$

b) $(p, q, r, s, p, q, r, s, ...)$ *is a solution of equation (1) of period four.*

c) $p.r = 0$ *and* $q.s = 0$.

d) *If there exists* $n_0 \in N$ *such that* $x_{n-1} \geq x_{n+1}$ *for all* $n \geq n_0$*, then*

$$\lim_{n\to\infty} x_n = 0.$$

e) *The following formulas*

$$x_{4n+1} = x_{-3}\left(1 - \frac{x_{-1}}{1+x_{-1}} \sum_{j=0}^{n} \prod_{i=1}^{2j} \frac{1}{1+x_{2i-1}}\right)$$

$$x_{4n+2} = x_{-2}\left(1 - \frac{x_0}{1+x_0} \sum_{j=0}^{n} \prod_{i=1}^{2j} \frac{1}{1+x_{2i}}\right)$$

$$x_{4n+3} = x_{-1}\left(1 - \frac{x_{-3}}{1+x_{-1}} \sum_{j=0}^{n} \prod_{i=1}^{2j+1} \frac{1}{1+x_{2i-1}}\right)$$

$$x_{4n+4} = x_0\left(1 - \frac{x_{-2}}{1+x_0} \sum_{j=0}^{n} \prod_{i=1}^{2j+1} \frac{1}{1+x_{2i}}\right)$$

hold.

f)*If* $x_{4n+1} \to p \neq 0$ *then* $x_{4n+3} \to 0$ *as* $n \to \infty$ *and if* $x_{4n+2} \to q \neq 0$ *then* $x_{4n+4} \to 0$ *as* $n \to \infty$.

Proof. **a)** Firstly, we consider the equation (1). From this equation we obtain

$$x_{n+1}(1 + x_{n-1}) = x_{n-3}.$$

If $x_{n-1} \in (0, +\infty)$, then $(1 + x_{n-1}) \in (1, +\infty)$. Since $x_{n+1} < x_{n-3}$, $n \in N$, we obtain that there exist $\lim\limits_{n\to\infty} x_{4n-3} = p$, $\lim\limits_{n\to\infty} x_{4n-2} = q$, $\lim\limits_{n\to\infty} x_{4n-1} = r$ and $\lim\limits_{n\to\infty} x_{4n} = s$.

b) $(p, q, r, s, p, q, r, s, ...)$ is a solution of equation (1) of period four.

c) In view of the equation (1), we obtain

$$x_{4n+1} = \frac{x_{4n-3}}{1 + x_{4n-1}}.$$

Take the limits on both sides of the above equality

$$\lim_{n\to\infty} x_{4n+1} = \lim_{n\to\infty} \frac{x_{4n-3}}{1 + x_{4n-1}}$$

then

$$p = \frac{p}{1 + r} \Rightarrow p + pr = p \Rightarrow p.r = 0.$$

Also, we obtain

$$x_{4n+2} = \frac{x_{4n-2}}{1 + x_{4n}}.$$

Take the limits on both sides of the above equality

$$\lim_{n\to\infty} x_{4n+2} = \lim_{n\to\infty} \frac{x_{4n-2}}{1 + x_{4n}}$$

then

$$q = \frac{q}{1 + s} \Rightarrow q + qs = q \Rightarrow q.s = 0.$$

d) If there exists $n_0 \in N$ such that $x_{n-1} \geq x_{n+1}$ for all $n \geq n_0$, then $p \leq r \leq p$ and $q \leq s \leq q$. Since $p.r = 0$ and $q.s = 0$ we obtain the result.

e) Subracting x_{n-3} from the left and right-hand sides in equation (1) we obtain

$$x_{n+1}-x_{n-3}=\frac{1}{1+x_{n-1}}(x_{n-1}-x_{n-5})$$

and the following formula

$$n\geq 2 for\begin{cases} x_{2n-3}-x_{2n-7}=(x_1-x_{-3})\prod\limits_{i=1}^{n-2}\frac{1}{1+x_{2i-1}} \\ x_{2n-2}-x_{2n-6}=(x_2-x_{-2})\prod\limits_{i=1}^{n-2}\frac{1}{1+x_{2i}} \end{cases} \tag{2}$$

holds. Replacing n by $2j$ in (2) and summing from $j=0$ to $j=n$ we obtain

$$x_{4n+1}-x_{-3}=(x_1-x_{-3})\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{2i-1}}\quad (n=0,1,2,...). \tag{3}$$

and

$$x_{4n+2}-x_{-2}=(x_2-x_{-2})\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{2i}}\quad (n=0,1,2,...). \tag{4}$$

Also, replacing n by $2j+1$ in (2) and summing from $j=0$ to $j=n$ we obtain

$$x_{4n+3}-x_{-1}=(x_1-x_{-3})\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{2i-1}}\quad (n=0,1,2,...) \tag{5}$$

and

$$x_{4n+4}-x_{0}=(x_2-x_{-2})\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{2i}}\quad (n=0,1,2,...) \tag{6}$$

Now, we obtained of the above formulas;

$$x_{4n+1}=x_{-3}(1-\frac{x_{-1}}{1+x_{-1}}\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{2i-1}}) \tag{7}$$

$$x_{4n+2}=x_{-2}(1-\frac{x_{0}}{1+x_{0}}\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{2i}}) \tag{8}$$

$$x_{4n+3}=x_{-1}(1-\frac{x_{-3}}{1+x_{-1}}\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{2i-1}}) \tag{9}$$

$$x_{4n+4} = x_0(1 - \frac{x_{-2}}{1+x_0}\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{2i}}) \tag{10}$$

f) Suppose that $p = r = 0$. By (e) we have

$$\lim_{n\to\infty} x_{4n+1} = \lim_{n\to\infty} x_{-3}(1 - \frac{x_{-1}}{1+x_{-1}}\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{2i-1}})$$

$$p = x_{-3}(1 - \frac{x_{-1}}{1+x_{-1}}\sum_{j=0}^{\infty}\prod_{i=1}^{2j}\frac{1}{1+x_{2i-1}})$$

$$p = 0 \Rightarrow \frac{1+x_{-1}}{x_{-1}} = \sum_{j=0}^{\infty}\prod_{i=1}^{2j}\frac{1}{1+x_{2i-1}}$$

Similarly,

$$\lim_{n\to\infty} x_{4n+3} = \lim_{n\to\infty} x_{-1}(1 - \frac{x_{-3}}{1+x_{-1}}\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{2i-1}})$$

$$r = x_{-1}(1 - \frac{x_{-3}}{1+x_{-1}}\sum_{j=0}^{\infty}\prod_{i=1}^{2j+1}\frac{1}{1+x_{2i-1}})$$

$$r = 0 \Rightarrow \frac{1+x_{-1}}{x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{2j+1}\frac{1}{1+x_{2i-1}} \tag{12}$$

From the equation (11) and (12),

$$\frac{1+x_{-1}}{x_{-1}} = \sum_{j=0}^{\infty}\prod_{i=1}^{2j}\frac{1}{1+x_{2i-1}} > \frac{1+x_{-1}}{x_{-3}} = \sum_{j=0}^{\infty}\prod_{i=1}^{2j+1}\frac{1}{1+x_{2i-1}} \tag{13}$$

thus, $x_{-3} > x_{-1}$.We arrive at a conradiction.
Suppose that $q = s = 0$. From that the equation (14) in (e) follows. Proof of the equation (13) is similar and will be omitted.

$$\frac{1+x_0}{x_0} = \sum_{j=0}^{\infty}\prod_{i=1}^{2j}\frac{1}{1+x_{2i}} > \frac{1+x_0}{x_{-2}} = \sum_{j=0}^{\infty}\prod_{i=1}^{2j}\frac{1}{1+x_{2i-1}} \tag{14}$$

thus, $x_{-2} > x_0$.We arrive at a conradiction which completes the proof of theorem. ∎

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