

The measure for a family of hypersurfaces with multiple components

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Abstract

In the real affine 5-dimensional space \mathbf{A}^5 , we show the measurability of the family of reducible hypersurfaces of type $\mathbf{S} = \mathbf{p}_1^{n_1} \cdot \dots \cdot \mathbf{p}_5^{n_5}$, where the components \mathbf{p}_i are hyperplanes passing through a fixed point and multiplicity $n_i \in \mathbf{N}$.

Mathematics Subject Classification: 60D05, 52A22.

Keywords: Geometric probability; stochastic geometry; random sets; random convex sets and integral geometry.

1 Introduction

Problems of measurability in the affine spaces have been studied by many authors, for various families of varieties and different kinds of geometric configurations (see [1],[2] and [5],[6]). Let \mathbf{X}_n be a space of dimension n , for example the affine or the projective space over the real field and

$$\mathbf{G}_r : x'_i = f_i(x_1, \dots, x_n; a_1, \dots, a_r), \quad (i = 1, \dots, n), \quad (1)$$

a Lie group of transformations on \mathbf{X}_n with a_1, \dots, a_r independent set of essential parameters for \mathbf{G}_r .

We assume that the identity of the group \mathbf{G}_r is determined by $a_1 = \dots = a_r = 0$. A function ϕ , solution of the system of partial differential equations

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} (\xi_j^i(x_1, \dots, x_n) \phi(x_1, \dots, x_n)) = 0, \tag{2}$$

where

$$\xi_j^i(x_1, \dots, x_n) = \left. \frac{\partial x'_i}{\partial a_j} \right|_{\mathbf{a}=\mathbf{0}}, \quad (j = 1, \dots, r),$$

is called invariant integral function for the group \mathbf{G}_r . M. Stoka [8] defines measurable a group \mathbf{G}_r with a unique invariant integral function ϕ , up to a constant factor. Let \mathcal{V}_q be a family of p -dimensional varieties defined by $\mathcal{V}_q : F_1(x_1, \dots, x_n; \alpha_1, \dots, \alpha_q) = 0, \dots, F_{n-q}(x_1, \dots, x_n; \alpha_1, \dots, \alpha_q) = 0$ where $\alpha_s \in \mathbf{R}$, $s = 1, \dots, q$ are essential independent parameters and $F_k, k = 1, \dots, (n - p)$ are analytic functions on its domains. Following the definitions and the proof in [8], if \mathbf{G}_r is the maximal invariant group of \mathcal{V}_q one has an associated group of transformations

$$\mathbf{H}_r : \alpha'_h = \varphi_h(\alpha_1, \dots, \alpha_q; a_1, \dots, a_r), \quad (h = 1, \dots, q),$$

and \mathbf{G}_r and \mathbf{H}_r are isomorphic.

If we suppose that \mathbf{H}_r is measurable of invariant integral function $\phi(\alpha_1, \dots, \alpha_q)$, the expressions

$$\mu_{\mathbf{G}_r}(\mathcal{V}_q) = \int_{\mathcal{F}(\alpha)} \phi(\alpha_1, \dots, \alpha_q) d\alpha_1 \wedge \dots \wedge d\alpha_q,$$

where $\mathcal{F}(\alpha) = \{(\alpha_1, \dots, \alpha_q) \in \mathbf{R}^q \mid F_k(\mathbf{x}; \alpha_1, \dots, \alpha_q) = 0, k = 1, \dots, n - p\}$ and

$$|\phi(\alpha_1, \dots, \alpha_q)| d\alpha_1 \wedge \dots \wedge d\alpha_q,$$

are called respectively the measure of the family \mathcal{V}_q and the invariant density respect to the group \mathbf{G}_r of the family \mathcal{V}_q . Hence, by definition, \mathcal{V}_q is measurable if there exists a unique non-zero function ϕ . Now let \mathbf{A}^5 be the five dimensional affine space (over the real field) of coordinates x_1, \dots, x_5 . The goal of this paper is to show the measurability of the family of reducible hypersurfaces in \mathbf{A}^5 of type

$$\mathbf{S} = \mathbf{p}_1^{n_1} \cdot \dots \cdot \mathbf{p}_5^{n_5},$$

where the components \mathbf{p}_i are hyperplanes passing trough a fixed point and multiplicity $n_i \in \mathbf{N}$.

2 Main Results

Without loss of generality, we can suppose that the fixed point for the components of the surface \mathbf{S} is the origin of coordinates. Hence the “generic” element of the family is the surface \mathbf{S} of equation

$$\mathbf{S} : \prod_{j=1}^5 \left(x_1 + A_j^{(1)} x_2 + \dots + A_j^{(4)} x_5 \right)^{n_j} = 0, \quad (n_j \in \mathbf{N}),$$

where $\det(A_j^{(\ell)}) \neq 0$, with $\ell = 0, \dots, 4$; $j = 1, \dots, 5$ and $A_j^{(0)} := 1$, for $1 \leq j \leq 5$.

The parameters space of this family is of dimension 20 and coordinates $A_j^{(i)} \in \mathbf{R}$ with $1 \leq i \leq 4$ and $j = 1, \dots, 5$. The maximal group of invariance of the family is

$$\mathbf{G}_{5^2} : x_r = \sum_{s=1}^5 \alpha_r^s x'_s, \quad r = 1, 2, \dots, 5 \tag{3}$$

with $\alpha_r^s \in \mathbf{R}$ and $\det(\alpha_r^s) \neq 0$.

Acting by \mathbf{G}_{5^2} on \mathbf{S} we obtain

$$\mathbf{S}' : \prod_{j=1}^5 \left(x_1 + [A_j^{(1)}]' x_2 + \dots + [A_j^{(4)}]' x_5 \right)^{n_j} = 0, \quad (n_j \in \mathbf{N}),$$

$[A_j^{(i)}]' \in \mathbf{R}$, with

$$\mathbf{H}_{5^2} : [A_j^{(i)}]' = \frac{\alpha_1^j + \sum_{l=1}^4 A_j^{(l)} \alpha_l^j}{\alpha_1^1 + \sum_{l=1}^4 A_j^{(l)} \alpha_l^1}, \tag{4}$$

where $1 \leq i \leq 4$; $j = 1, \dots, 5$ and $\alpha_1^1 + \sum_{l=1}^4 A_j^{(l)} \alpha_l^1 \neq 0, \forall j = 1, \dots, 5$.

Theorem 2.1 *The family of reducible hypersurfaces in \mathbf{A}^5 of “generic” element of equation*

$$\mathbf{S} : \prod_{j=1}^5 \left(x_1 + A_j^{(1)} x_2 + \dots + A_j^{(4)} x_5 \right)^{n_j} = 0, \quad (n_j \in \mathbf{N}),$$

where $\det(A_j^{(\ell)}) \neq 0$ ($\ell = 0, \dots, 4$; $j = 1, \dots, 5$, $A_j^{(0)} := 1$, for $1 \leq j \leq 5$) is measurable and its unique invariant integral function is given by

$$\phi = \frac{1}{\left(\det A_i^{(\ell)} \right)^5}.$$

Proof: By direct calculations we give the following Deltheil's system:

$$\sum_{j=1}^5 \sum_{i=1}^4 A_j^{(i)} \frac{\partial \phi}{\partial A_j^{(i)}} = -20\phi,$$

$$\sum_{j=1}^5 \frac{\partial \phi}{\partial A_j^{(i)}} = 0, \quad i = 1, \dots, 4$$

$$\sum_{j=1}^5 A_j^{(i)} \frac{\partial \phi}{\partial A_j^{(i)}} = -5\phi,$$

$$\sum_{j=1}^5 A_j^{(i)} \frac{\partial \phi}{\partial A_j^{(k)}} = 0, \quad i, k = 1, \dots, 4; \quad k \neq i.$$

And finally

$$\sum_{j=1}^5 \left(A_j^{(h)} \right)^2 \frac{\partial \phi}{\partial A_j^{(h)}} + \sum_{j=1}^5 A_j^{(h)} A_j^{(k)} \frac{\partial \phi}{\partial A_j^{(k)}} = -5 \sum_{j=1}^5 A_j^{(h)} \phi,$$

$$\sum_{j=1}^5 \left(A_j^{(k)} \right)^2 \frac{\partial \phi}{\partial A_j^{(k)}} + \sum_{j=1}^5 A_j^{(h)} A_j^{(k)} \frac{\partial \phi}{\partial A_j^{(h)}} = -5 \sum_{j=1}^5 A_j^{(k)} \phi,$$

for every pair (h, k) with $h \neq k = 1, \dots, 4$.

The unique non-zero solution (up to a constant factor) is

$$\phi = \frac{1}{\left(\det A_i^{(\ell)} \right)^5}.$$

Remark The three dimensional and reduced case is investigated by Santoro in [5].

References

- [1] P. Dulio, M. Petriccione, Sulla misurabilità delle famiglie dei sistemi di k iperpiani passanti per un punto fisso dello spazio affine A_7 , *Atti Acc. Peloritana dei Pericolanti Classe S.M.F.N.* Vol.LXXI (1993).
- [2] P. Dulio, M. Petriccione, Sulla misurabilità delle famiglie dei sistemi di due e tre iperpiani passanti per un punto fisso di A_4 , *Atti Acc. Peloritana dei Pericolanti Classe S.M.F.N.* Vol.LXXI (1993).

- [3] A. Duma, M. Stoka, On the measurability of the conic sections' family in the projective space \mathbf{P}^3 , *Rend. Circ. Mat. Palermo*, Serie II, Tomo LI (2002), 537-548.
- [4] G. Failla, G. Molica Bisci, On the measurability of the conic sections family in the projective space \mathbf{P}^4 , to appear *Boll. U.M.I.*
- [5] G. Santoro, Sulla misurabilità della famiglia di superfici cubiche spezzate in tre piani passanti per un punto fisso, *Atti Acc. Peloritana dei Pericolanti Classe S.M.F.N.*, Vol.LXVIII, (1990).
- [6] G. Santoro, Sulla misurabilità di una famiglia di superfici cubiche dello spazio affine A_3 , *Atti Acc. Peloritana dei Pericolanti Classe S.M.F.N.*, Vol.LXVIII, (1990).
- [7] M. Stoka, Masura familiilor de varietati din spatiul E_3 , *St. Cerc. Mat.*, IX, (1958),547-558.
- [8] M. Stoka, *Géométrie Intégrale*, Mém. Sci. Math., fasc. 165, Gauthier-Villars, Paris, 1968.
- [9] M. Stoka, On the measurability of the conic sections' family in the projective space \mathbf{P}^n , *Boll. U.M.I.* (8) 7-B (2004), 357-367.

Received: June 6, 2006