Ore extensions over $\delta$-rigid rings

V. K. Bhat and Ravi Raina

School of Applied Physics and Mathematics, SMVD University
P/o Kakryal, Udhampur, J and K, India - 182121
vijaykumarbhat2000@yahoo.com

Om Prakash

Department of Mathematics Banasthali Vidyapith
Rajasthan, India - 304022

Abstract. In this article, we find a relation between the prime radical of a 2-primal ring $R$ and that of $R[x, \sigma, \delta]$, where $\sigma$ is an automorphism of $R$ and $\delta$ is a $\sigma$-derivation of $R$.

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1. Introduction

A ring $R$ always means an associative ring. $Q$ denotes the field of rational numbers. MinSpec($R$) denotes the sets of all minimal prime ideals of $R$. P($R$) and N($R$) denote the prime radical and the set of all nilpotent elements of $R$ respectively. Recall that $R[x, \sigma, \delta]$ is the usual polynomial ring with coefficients in $R$ and we consider any $f(x) \in R[x, \sigma, \delta]$ to be of the form $f(x) = \sum x^i a_i$, $0 \leq i \leq n$. Multiplication in $R[x, \sigma, \delta]$ is subject to the relation $ax = x\sigma(a) + \delta(a)$ for $a \in R$. In this article, we discuss completely prime ideals and the prime radical of a 2-primal ring and try to relate completely prime ideals of a ring $R$ with the completely prime ideals of $R[x, \sigma, \delta]$. This is given in 2.4. We also find a relation between the prime radical of a 2-primal ring $R$ and that of $R[x, \sigma, \delta]$. This is given in 2.6. Recall that a ring $R$ is 2-primal if $N(R) = P(R)$. $R$ is 2-primal if and only if $P(R)$ is completely semiprime (i.e. $a^2 \in P(R)$ implies $a \in P(R)$, $a \in R$). We also note that any reduced ring is 2-primal, and any commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [3, 5, 7, 10].
Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [1, 2, 4, 8, 9]. In this article we deal with a \(\sigma\)-derivation of a ring \(R\). Let \(R\) be a ring. Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) be a \(\sigma\)-derivation of \(R\). We define a \(\delta\)-rigid ring 2.1, and establish a relation between a \(\delta\)-rigid ring and a 2-primal ring. We also find a relation between the prime radical of a \(\delta\)-rigid ring \(R\) and that of \(R[x, \sigma, \delta]\). Recall that an ideal \(I\) of a ring \(R\) is called \(\sigma\)-invariant if \(\sigma(I) = I\) and is called \(\delta\)-invariant if \(\delta(I) \subseteq I\). Also \(I\) is called completely prime if \(ab \in I\) implies \(a \in I\) or \(b \in I\) for \(a, b \in R\).

2. Main Result

We begin with the following definition:

**Definition 2.1.** Let \(R\) be a ring. Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) be a \(\sigma\)-derivation of \(R\). We say that \(R\) is a \(\delta\)-rigid ring if \(a\delta(a) = 0\) implies \(a = 0\), \(a \in R\). We note that a ring \(R\) with identity \(1\) is not a \(\delta\)-rigid ring as \(1\delta(1) = 0\).

**Proposition 2.2.** Let \(R\) be a 2-primal ring. Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) be a \(\sigma\)-derivation of \(R\) such that \(\delta(P(R)) \subseteq P(R)\). Let \(P \in \text{MinSpec}(R)\) be such that \(\sigma(P) = P\). Then \(\delta(P) \subseteq P\).

**Proof.** The proof follows from Theorem (3.6) and Lemma (3.2) of [6]. We give a sketch of the proof.

Let \(P \in \text{MinSpec}(R)\) with \(\sigma(P) = P\). Let \(a \in P\). Then there exists \(b \notin P\) such that \(ab \in P\) by Corollary (1.10) of [9]. Now we have \(\delta(P(R)) \subseteq P(R)\). Therefore \(\delta(ab) = \delta(a)(\sigma(b) + a\sigma(b)) \in P\). So we have \(\delta(a)(\sigma(b) + a\sigma(b)) \in P\). But \(\sigma(b) \notin P\), and therefore \(\delta(a) \in P\) as by Proposition (1.11) of [9] \(P\) is completely prime. Hence \(\delta(P) \subseteq P\). \(\square\)

**Theorem 2.3.** Let \(R\) be a \(\delta\)-rigid ring. Let \(\sigma\) be an automorphism of \(R\) such that \(\sigma(P(R)) = P(R)\), and \(\delta\) be a \(\sigma\)-derivation of \(R\) such that \(\delta(P(R)) \subseteq P(R)\). Then \(R\) is 2-primal.

**Proof.** Define a map \(\partial : R/P(R) \to R/P(R)\) by \(\partial(a + P(R)) = \delta(a) + P(R)\) for \(a \in R\) and \(\tau : R/P(R) \to R/P(R)\) a map by \(\tau(a + P(R)) = \sigma(a) + P(R)\) for \(a \in R\). Now it is easy to see that that \(\tau\) is an automorphism of \(R/P(R)\). Also for any \(a + P(R), b + P(R) \in R/P(R)\); \(\partial((a + P(R))(b + P(R))) = \partial(ab + P(R)) = \delta(ab) + P(R) = \delta(a)\sigma(b) + a\delta(b) + P(R) = (\delta(a) + P(R))(\sigma(b) + P(R)) + (a + P(R))(\delta(b) + P(R)) = \delta(a + P(R))\tau(b + P(R)) + (a + P(R))\partial(b + P(R))\), and it is obvious that \(\partial(a + P(R) + b + P(R)) = \partial(a + P(R)) + \partial(b + P(R))\). Therefore \(\partial\) is a \(\tau\)-derivation of \(R/P(R)\). Now a \(\delta(a) = 0\) if and only if \((a + P(R))\partial(a + P(R)) = P(R)\) in \(R/P(R)\). Thus, as in Proposition (5) of [4], \(R\) is a reduced ring and hence \(R\) is 2-primal. \(\square\)

**Proposition 2.4.** Let \(R\) be a ring. Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) be a \(\sigma\)-derivation of \(R\). Then:
1. For any completely prime ideal $P$ of $R$ with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.

2. For any completely prime ideal $Q$ of $R[x, \sigma, \delta]$, $Q \cap R$ is a completely prime ideal of $R$.

Proof. (1) Let $P$ be a completely prime ideal of $R$. Now let $f(x) = \sum x^i a_i \in R[x, \sigma, \delta]$ and $g(x) = \sum x^j b_j \in R[x, \sigma, \delta]$, $0 \leq i \leq n$, $0 \leq j \leq m$ such that $f(x)g(x) \in P[x, \sigma, \delta]$. Suppose $f(x) \notin P[x, \sigma, \delta]$. We will show that $g(x) \in P[x, \sigma, \delta]$. Suppose $f(x) \notin P[x, \sigma, \delta]$. We use induction on $n$ and $m$. For $n = m = 1$, the verification is easy. We check for $n = 2$ and $m = 1$. Let $f(x) = x^2 a + xb + c$ and $g(x) = xu + v$. Now $f(x)g(x) \in P[x, \sigma, \delta]$ with $f(x) \notin P[x, \sigma, \delta]$. The possibilities are $a \notin P$ or $b \notin P$ or $c \notin P$ or any two out of these three do not belong to $P$ or all of them do not belong to $P$. We verify case by case.

Let $a \notin P$. Since $x^3 \sigma(a) + x^2 \delta(a)u + \sigma(b)u + av + x(\delta(b)u + \sigma(c)u + bv) + \delta(c)u + cv \in P[x, \sigma, \delta]$, we have $\sigma(a)u \in P$, and so $u \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies $av \in P$, and so $v \in P$. Therefore $g(x) \in P[x, \sigma, \delta]$.

Let $b \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then $\sigma(a) \in P$ and therefore $a, \sigma(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $(b)u \in P$ which in turn implies that $b \in P$, which is not the case. Therefore we have $u \in P$. Now $(b)u + (c)u + bv \in P$ implies that $bv \in P$ and therefore $v \in P$. Thus we have $g(x) \in P[x, \sigma, \delta]$.

Let $c \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then as above $a, \sigma(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$. Now $u \notin P$ implies that $\sigma(b) \in P$; i.e. $b, \delta(b) \in P$. Also $\delta(b)u + \sigma(c)u + bv \in P$ implies $\sigma(c)u \in P$ and therefore $\sigma(c) \in P$ which is not the case. Thus we have $u \in P$. Now $\delta(c)u + cv \in P$ implies $cv \in P$, and so $v \in P$. Therefore $g(x) \in P[x, \sigma, \delta]$. The remaining cases are now obvious. Using the same arguments, the result can be verified for $n \geq 3$ and $m \geq 2$ also.

(2) Let $Q$ be a completely prime ideal of $R[x, \sigma, \delta]$. Suppose $a, b \in R$ are such that $ab \in Q \cap R$ with $a \notin Q \cap R$. This means that $a \notin Q$ as $a \in R$. Thus we have $ab \in Q \cap R \subseteq Q$, with $a \notin Q$. Therefore we have $b \in Q$, and thus $b \in Q \cap R$.

The above discussion leads to the following question:

Is $\delta(Q \cap R) \subseteq Q \cap R$ in 2.4? If so, is $Q = (Q \cap R)[x, \sigma, \delta]$? The question remains to be answered, but in this connection we note that $\sigma$ and $\delta$ can be extended to $R[x, \sigma, \delta]$ by taking $\sigma(x) = x$ and $\delta(x) = 0$. In other words, $\sigma(xa) = x\sigma(a)$ and $\delta(xa) = x\delta(a)$ for all $a \in R$.

Corollary 2.5. Let $R$ be a $\delta$-rigid ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$ such that $\delta(P(R)) \subseteq P(R)$. Let $P \in \text{MinSpec}(R)$ be such that $\sigma(P) = P$. Then $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.

Proof. $R$ is 2-primal by 2.3, and so by 2.2 $\delta(P) \subseteq P$. Further more $P$ is a completely prime ideal of $R$ by Proposition (1.11) of [9]. Now use 2.4. 

Theorem 2.6. Let $R$ be a $δ$-rigid ring. Let $σ$ be an automorphism of $R$ and $δ$ be a $σ$-derivation of $R$ such that $δ(P(R)) ⊆ P(R)$ and $σ(P) = P$ for all $P ∈ \text{MinSpec}(R)$. Then $R[x, σ, δ]$ is 2-primal if and only if $P(R)[x, σ, δ] = P(R[x, σ, δ])$.

Proof. Let $R[x, σ, δ]$ be 2-primal. Let $P ∈ \text{MinSpec}(R)$. By 2.5 $P[x, σ, δ]$ is a completely prime ideal of $R[x, σ, δ]$, and therefore $P(R[x, σ, δ]) ⊆ P(R)R[x, σ, δ]$. One may see Proposition (3.8) of [6] also. Let $f(x) = \sum x^i a_i ∈ P(R)[x, σ, δ]$, $0 ≤ i ≤ n$. Now $R$ is a 2-primal subring of $R[x, σ, δ]$ by 2.3. This implies that $a_j$ is nilpotent and thus $a_j ∈ N(R[x, σ, δ]) = P(R[x, σ, δ])$, and so we have $x^j a_j ∈ P(R[x, σ, δ])$ for each $j$. Therefore $f(x) ∈ P(R[x, σ, δ])$. Hence we have $P(R)[x, σ, δ] = P(R[x, σ, δ])$.

Conversely suppose $P(R)[x, σ, δ] = P(R[x, σ, δ])$. We will show that $R[x, σ, δ]$ is 2-primal. Let $g(x) = \sum x^i b_i ∈ R[x, σ, δ]$, $0 ≤ i ≤ n$ be such that $(g(x))^2 ∈ P(R[x, σ, δ]) = P(R)[x, σ, δ]$. Then by an easy induction and by using the fact that $P(R)$ is completely semiprime by 2.3, it can be easily seen that $b_i ∈ P(R)$ for all $b_i$, $0 ≤ i ≤ n$. This means that $f(x) ∈ P(R)[x, σ, δ] = P(R[x, σ, δ])$. Therefore $P(R[x, σ, δ]$ is completely semiprime. Hence $R[x, σ, δ]$ is 2-primal.

We now have some examples:

1. Let $R$ be a Noetherian $Q$-algebra satisfying the conditions of 2.6. Then $R[x, σ, δ]$ is 2-primal.

2. Consider $R = Z_2 ⊕ Z_2$. Then $R$ is a commutative reduced ring. Define a map $σ : R → R$ by $σ(a, b) = (b, a)$. Then $σ$ is an automorphism of $R$. Now define a map $δ : R → R$ by $δ(a, b) = (a-b, 0)$. Then $δ$ is a $σ$-derivation of $R$. But $R$ is not a $δ$-rigid ring, as $(0, 1)δ(0, 1) = (0, 0)$.

3. Consider $R = (a_{ij})_{2,2}$, the set of all $2x2$ matrices over the ring $nZ$, $n > 1$ with $a_{21} = 0$. Define $σ : R → R$ by $σ(a_{ij}) = (b_{ij})$, where $b_{ij} = a_{ij}$ except that $b_{12} = -a_{12}$. Then it can be seen that $δ$ is an automorphism of $R$. Now define $δ : R → R$ by $δ(a_{ij}) = (c_{ij})$, where $c_{ij} = 0$ except that $c_{12} = 2a_{12} + a_{22} - a_{11}$. Then it can be seen that $δ$ is a $σ$-derivation of $R$. But $R$ is not a $δ$-rigid ring, as for $A = (a_{ij})_{2,2}$, with $a_{ij} = 0$ except $a_{22} = 1$, $Aδ(A) = (0)$.

We finally have the following:

Remark 2.7. If $σ$ is identity map, we get these results for the differential operator ring $R[x, δ]$, and if $δ$ is zero map, we get these results for the skew-polynomial ring $R[x, σ]$.

References


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