Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 12, 591 - 595

Ore extensions over δ-rigid rings

V. K. Bhat and Ravi Raina

School of Applied Physics and Mathematics, SMVD University
P/o Kakryal, Udhampur, J and K, India - 182121
vijaykumarbhat2000@yahoo.com

Om Prakash

Department of Mathematics Banasthali Vidyapith
Rajasthan, India - 304022

Abstract. In this article, we find a relation between the prime radical of a 2-primal ring R and that of $R[x, \sigma, \delta]$, where σ is an automorphism of R and δ is a σ-derivation of R.

Mathematics Subject Classification: Primary 16-XX; Secondary 16P40, 16P50, 16U20

Keywords: Radical, automorphism, derivation, completely prime, δ-ring

1. Introduction

A ring R always means an associative ring. Q denotes the field of rational numbers. MinSpec(R) denotes the sets of all minimal prime ideals of R. P(R) and N(R) denote the prime radical and the set of all nilpotent elements of R respectively. Recall that $R[x, \sigma, \delta]$ is the usual polynomial ring with coefficients in R and we consider any $f(x) \in R[x, \sigma, \delta]$ to be of the form f(x) = $\sum x^i a_i$, $0 \leq i \leq n$. Multiplication in $R[x, \sigma, \delta]$ is subject to the relation ax = $x\sigma(a) + \delta(a)$ for $a \in R$. In this article, we discuss completely prime ideals and the prime radical of a 2-primal ring and try to relate completely prime ideals of a ring R with the completely prime ideals of $R[x, \sigma, \delta]$. This is given in 2.4. We also find a relation between the prime radical of a 2-primal ring R and that of $R[x, \sigma, \delta]$. This is given in 2.6. Recall that a ring R is 2-primal if N(R) = P(R). R is 2-primal if and only if P(R) is completely semiprime (i.e. $a^2 \in P(R)$ implies $a \in P(R)$, $a \in R$). We also note that any reduced ring is 2-primal, and any commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [3, 5, 7, 10].

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [1, 2, 4, 8, 9]. In this article we deal with a σ-derivation of a ring R. Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. We define a δ-rigid ring 2.1, and establish a relation between a δ-rigid ring and a 2-primal ring. We also find a relation between the prime radical of a δ-rigid ring R and that of $R[x, \sigma, \delta]$. Recall that an ideal I of a ring R is called σ-invariant if $\sigma(I) =$ I and is called δ-invariant if $\delta(I) \subseteq I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$.

2. MAIN RESULT

We begin with the following definition:

Definition 2.1. Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. We say that R is a δ-rigid ring if $a\delta(a) = 0$ implies a = 0, $a \in R$. We note that a ring R with identity 1 is not a δ-rigid ring as $1\delta(1) = 0$

Proposition 2.2. *Let R be a 2-primal ring. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. Let $P \in MinSpec(R)$ be such that $\sigma(P) = P$. Then $\delta(P) \subseteq P$.*

Proof. The proof follows from Theorem (3.6) and Lemma (3.2) of [6]. We give a sketch of the proof.

Let $P \in MinSpec(R)$ with $\sigma(P) =$ P. Let $a \in P$. Then there exists $b \notin P$ such that $ab \in P(R)$ by Corollary (1.10) of [9]. Now we have $\delta(P(R)) \subseteq P(R)$. Therefore $\delta(ab) = \delta(a)\sigma(b) + a\sigma(b) \in P(R) \subseteq P$. So we have $\delta(a)\sigma(b) \in P$. But $\sigma(b) \notin P$, and therefore $\delta(a) \in P$ as by Proposition (1.11) of [9] P is completely prime. Hence $\delta(P) \subseteq P$. □

Theorem 2.3. *Let R be a δ-rigid ring. Let σ be an automorphism of R such that $\sigma(P(R)) = P(R)$, and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof. Define a map $\partial : R/P(R) \to R/P(R)$ by $\partial(a+P(R)) = \delta(a)+P(R)$ for $a \in R$ and $\tau : R/P(R) \to R/P(R)$ a map by $\tau(a + P(R)) = \sigma(a) + P(R)$ for $a \in R$. Now it is easy to see that that τ is an automorphism of $R/P(R)$. Also for any a + P(R), $b+P(R) \in R/P(R)$; $\partial((a+P(R))(b+P(R)) = \partial(ab+P(R))$ $= \delta(ab) + P(R) = \delta(a)\sigma(b) + a\delta(b) + P(R) = (\delta(a) + P(R))(\sigma(b) + P(R)) +$ $(a+P(R))(\delta(b)+P(R)) = \partial(a+P(R))\tau(b+P(R)) + (a+P(R))\partial(b+P(R))$, and it is obvious that $\partial(a + P(R) + b + P(R)) = \partial(a + P(R)) + \partial(b + P(R))$. Therefore ∂ is a $\tau - derivation$ of R/P(R). Now a $\delta(a) = 0$ if and only if $(a + P(R))\partial(a + P(R)) =$ P(R) in R/P(R). Thus, as in Proposition (5) of [4], R is a reduced ring and hence R is 2-primal. □

Proposition 2.4. *Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. Then:*

1. *For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $P[x,\sigma,\delta]$ is a completely prime ideal of $R[x,\sigma,\delta]$.*
2. *For any completely prime ideal Q of $R[x,\sigma,\delta]$, $Q\cap R$ is a completely prime ideal of R.*

Proof. **(1)** Let P be a completely prime ideal of R. Now let f(x) = $\sum x^i a_i \in R[x,\sigma,\delta]$ and g(x) = $\sum x^j b_j \in R[x,\sigma,\delta]$, $0 \le i \le n$, $0 \le j \le m$ such that $f(x)g(x) \in P[x,\sigma,\delta]$. Suppose $f(x) \notin P[x,\sigma,\delta]$. We will show that $g(x) \in P[x,\sigma,\delta]$. We use induction on n and m. For n = m = 1, the verification is easy. We check for n = 2 and m = 1. Let f(x) = $x^2a + xb + c$ and g(x) = xu + v. Now $f(x)g(x) \in P[x,\sigma,\delta]$ with $f(x) \notin P[x,\sigma,\delta]$. The possibilities are $a \notin P$ or $b \notin P$ or $c \notin P$ or any two out of these three do not belong to P or all of them do not belong to P. We verify case by case.

Let $a \notin P$. Since $x^3\sigma(a)u + x^2(\delta(a)u + \sigma(b)u + av) + x(\delta(b)u + \sigma(c)u + bv) + \delta(c)u + cv \in P[x,\sigma,\delta]$, we have $\sigma(a)u \in P$, and so $u \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies $av \in P$, and so $v \in P$. Therefore $g(x) \in P[x,\sigma,\delta]$.

Let $b \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then $\sigma(a) \in P$ and therefore a, $\delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $(b)u \in P$ which in turn implies that $b \in P$, which is not the case. Therefore we have $u \in P$. Now $(b)u + (c)u + bv \in P$ implies that $bv \in P$ and therefore $v \in P$. Thus we have $g(x) \in P[x,\sigma,\delta]$.

Let $c \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then as above a, $\delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$. Now $u \notin P$ implies that $\sigma(b) \in P$; i.e. $b, \delta(b) \in P$. Also $\delta(b)u + \sigma(c)u + bv \in P$ implies $\sigma(c)u \in P$ and therefore $\sigma(c) \in P$ which is not the case. Thus we have $u \in P$. Now $\delta(c)u + cv \in P$implies $cv \in P$, and so $v \in P$. Therefore $g(x) \in P[x,\sigma,\delta]$. The remaining cases are now obvious. Using the same arguments, the result can be verified for $n \ge 3$ and $m \ge 2$ also.

(2) Let Q be a completely prime ideal of $R[x,\sigma,\delta]$. Suppose $a, b \in R$ are such that $ab \in Q \cap R$ with $a \notin Q \cap R$. This means that $a \notin Q$ as $a \in R$. Thus we have $ab \in Q \cap R \subseteq Q$, with $a \notin Q$. Therefore we have $b \in Q$, and thus $b \in Q \cap R$. □

The above discussion leads to the following question:

Is $\delta(Q \cap R) \subseteq Q \cap R$ in 2.4? If so, is Q = $(Q \cap R)[x,\sigma,\delta]$? The question remains to be answered, but in this connection we note that σ and δ can be extended to $R[x,\sigma,\delta]$ by taking $\sigma(x)$ = x and $\delta(x) = 0$. In other words, $\sigma(xa)$ = $x\sigma(a)$ and $\delta(xa) = x\delta(a)$ for all $a \in R$.

Corollary 2.5. *Let R be a δ-rigid ring. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. Let $P \in MinSpec(R)$ be such that $\sigma(P) = P$. Then $P[x,\sigma,\delta]$ is a completely prime ideal of $R[x,\sigma,\delta]$.*

Proof. R is 2-primal by 2.3, and so by 2.2 $\delta(P) \subseteq P$. Further more P is a completely prime ideal of R by Proposition (1.11) of [9]. Now use 2.4. □

Theorem 2.6. *Let R be a $\delta-$ rigid ring. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $R[x, \sigma, \delta]$ is 2-primal if and only if $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.*

Proof. Let $R[x, \sigma, \delta]$ be 2-primal. Let $P \in MinSpec(R)$. By 2.5 $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$, and therefore $P(R[x, \sigma, \delta]) \subseteq P(R)R[x, \sigma, \delta]$. One may see Proposition (3.8) of [6] also. Let f(x) = $\sum x^j a_j \in P(R)[x, \sigma, \delta]$, $0 \leq i \leq n$. Now R is a 2-primal subring of $R[x, \sigma, \delta]$ by 2.3. This implies that a_j is nilpotent and thus $a_j \in N(R[x, \sigma, \delta]) = P(R[x, \sigma, \delta]$, and so we have $x^j a_j \in P(R[x, \sigma, \delta])$ for each j. Therefore $f(x) \in P(R[x, \sigma, \delta])$. Hence we have $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.

Conversely suppose $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$. We will show that $R[x, \sigma, \delta]$ is 2-primal. Let g(x) = $\sum x^i b_i \in R[x, \sigma, \delta]$, $0 \leq i \leq n$ be such that $(g(x))^2 \in P(R[x, \sigma, \delta]) = P(R)[x, \sigma, \delta]$. Then by an easy induction and by using the fact that P(R) is completely semiprime by 2.3, it can be easily seen that $b_i \in P(R)$ for all b_i, $0 \leq i \leq n$. This means that $f(x) \in P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$. Therefore $P(R[x, \sigma, \delta]$ is completely semiprime. Hence $R[x, \sigma, \delta]$ is 2-primal. □

We now have some examples:

1. Let R be a Noetherian Q-algebra satisfying the conditions of 2.6. Then $R[x, \sigma, \delta]$ is 2-primal.
2. Consider R = $Z_2 \oplus Z_2$. Then R is a commutative reduced ring. Define a map $\sigma : R \rightarrow Rby\sigma(a, b) =$ (b, a). Then σ is an automorphism of R. Now define a map $\delta : R \rightarrow R$ by $\delta(a, b)$= (a-b, 0). Then δ is a σ-derivation of R. But R is not a δ-rigid ring, as (0, 1)$\delta(0, 1)$ = (0, 0).
3. Consider R = $(a_{ij})_{2,2}$, the set of all 2x2 matrices over the ring nZ, $n > 1$ with $a_{21} = 0$. Define σ: R $\rightarrow R$ by $\sigma(a_{ij}) = (b_{ij})$, where $b_{ij} = a_{ij}$ except that $b_{12} = -a_{12}$. Then it can be seen that δ is an automorphism of R. Now define $\delta : R \rightarrow R$ by $\delta(a_{ij}) = (c_{ij})$, where $c_{ij} = 0$ except that $c_{12} = 2a_{12} + a_{22} - a_{11}$. Then it can be seen that δ is a σ-derivation of R. But R is not a δ-rigid ring, as for A = $(a_{ij})_{2,2}$, with a_{ij} =0 except $a_{22} = 1$, $A\delta(A) = (0)$.

We finally have the following:

Remark 2.7. If σ is identity map, we get these results for the differential operator ring $R[x, \delta]$, and if δ is zero map, we get these results for the skew-polynomial ring $R[x, \sigma]$.

REFERENCES

[1] S. Annin, Associated primes over skew polynomial rings. Communications in Algebra 30 (2002) 2511-2528.

[2] W.D. Blair and L.W. Small, Embedding differential and skew-polynomial rings into artinain rings. Proc. Amer. Math. Soc. 109(4) 1990, 881-886.

[3] C.Y. Hong and T.K. Kwak, On minimal strongly prime ideals. Communications in Algebra 28(10) (2000), 4868-4878.

[4] C.Y. Hong, N.K. Kim and T.K. Kwak, Ore-extensions of Baer and p.p.-rings. J.Pure and Applied Algebra 151(3) (2000), 215-226.

[5] C.Y. Hong, N.K. Kim, T.K. Kwak and Y. Lee, On weak -regularity of rings whose prime ideals are maximal. J.Pure and Applied Algebra 146(2000), 35-44.

[6] C.Y. Hong, T.K. Kwak, and S.T. Rizvi, Rigid ideals and radicals of Ore extensions. Algebra Colloquium 12(3) (2005), 399-412.

[7] N.K. Kim and T.K. Kwak, Minimal prime ideals in 2-primal rings. Math. Japonica 50(3) (1999), 415-420.

[8] J. Krempa, Some examples of reduced rings. Algebra Colloq.3: 4(1996), 289-300.

[9] T.K. Kwak, Prime radicals of skew-polynomial rings. Int. J. of Mathematical Sciences 2(2) (2003), 219-227.

[10] G.Y. Shin, Prime ideals and sheaf representations of a pseudo symmetric ring. Trans. Amer. Math. Soc. 184(1973), 43-60.

Received: May 16, 2006