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Construction of singular hypersurfaces and linkage over a finite field

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Abstract. Here we prove two existence theorem over $\mathbb{F}_q$: existence of hypersurfaces with prescribed isolated singularities and existence of "smooth" linkage.

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1. The statements

Here we consider two existence theorems over $\mathbb{F}_q$. The corresponding constructions are obvious over $\bar{F}_q$ and the aim is just to find a relatively low prime power q such that the same constructions may be done over $\mathbb{F}_q$. For any $P \in \mathbf{P}^n(\bar{F}_q)$ and any integer $m > 0$ let mP denote the infinitesimal neighborhood of order $m-1$ of P in $\mathbf{P}^n$. Set $0P = \emptyset$. In section 2 we will study the case of hypersurfaces with prescribed isolated singularities and prove the following result.

Theorem 1. *Fix a prime power q, an integer $n \geq 2$, an integer $d > 0$, an integer s such that $1 \leq s \leq (q^{n+1}-1)/(q-1)$, integers $m_i > 0$, and s distinct points $P_1, \dots, P_s \in \mathbf{P}^n(\mathbb{F}_q)$. Let $Z := \cup_{,i=1}^{s} m_i P_i$ and assume $h^1(\mathbf{P}^n, \mathcal{I}_Z(d-1)) = 0$. Set $\delta := d^n - \sum_{1=1}^{s} m_i^n$ and $\delta_i := m_i^{n-1}$. Assume $q \geq (\delta-1)\delta^n$. Then there exists a degree d hypersurface $X \subset \mathbf{P}^n$ defined over $\mathbb{F}^q$ and such that Sing$(X) \subseteq \{P_1, \dots, P_s\}$, $P_i \in$ Sing(X) if and only if $m_i \geq 2$, and X has multiplicity m_i at each P_i. Furthermore, if $q \geq (\delta-1)\delta^n + \sum_{i=1}^{s}(\delta_i - 1)\delta_i^{n-1}$, then we may find X such that X has an ordinary multiple point with multiplicity m_i at P_i, i.e. the tangent cone of X at P_i is a cone over a smooth degree m_i hypersurface of $\mathbf{P}^{n-1}$.*

When $P_1, \dots, P_s \in \mathbf{P}^n(\bar{F}_q)$, $P_i \notin \mathbf{P}^n(\mathbb{F}_q)$ for some i, but the set of all pairs $\{(P_1, m_1), \dots, (P_s, m_s)\}$ is invariant for the natural action of the absolute Galois group of $\mathbb{F}_q$ we are able to prove the following result.

Theorem 2. *Fix a prime power q, an integer $n \geq 2$, an integer $d > 0$, an integer s such that $1 \leq s \leq (q^{n+1}-1)/(q-1)$, integers $m_i > 0$, and s distinct points $P_1, \dots, P_s \in \mathbf{P}^n(\bar{\mathbb{F}}_q)$. Let $Z := \cup_{,i=1}^{s} m_i P_i$ and assume $h^1(\mathbf{P}^n, \mathcal{I}_Z(d-1)) = 0$. Assume that the scheme Z and the inclusion of Z in $\mathbf{PO}^n$ are defined over $\mathbb{F}_q$, i.e. assume that the absolute Galois group of $\mathbb{F}_q$ acts trivially on the set of pairs*

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$\{(P_1,m_1),\dots,(P_s,m_s)\}$. Set $\delta := d^n - \sum_{1=1}^{s} m_i^n$ and $\delta_i := m_i^{n-1}$. Assume $q \ge (\delta-1)\delta^n$. Then there exists a degree d hypersurface $X \subset \mathbf{P}^n$ defined over $\mathbb{F}^q$ and such that Sing$(X) \subseteq \{P_1,\dots,P_s\}$, $P_i \in$ Sing(X) if and only if $m_i \ge 2$, and X has multiplicity m_i at each P_i. Furthermore, if $q \ge (\delta-1)\delta^n + \sum_{i=1}^{s}(\delta_i - 1)\delta_i^{n-1}$, then we may find X such that X has an ordinary multiple point with multiplicity m_i at P_i, i.e. the tangent cone of X at P_i is a cone over a smooth degree m_i hypersurface of $\mathbf{P}^{n-1}$.

Remark 1. Take Z as in the statements of Theorems 1 and 2 and let μ be the first integer $t \ge -1$ such that $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) = 0$. Thus $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) = 0$ for all $t \ge \mu$ and $d \ge \mu + 1$. It is classical that $\mu \le m_1 + \cdots + m_s - 1$ and that we have equality if and only if the points $P_1,\dots,P_s$ are collinear ([3]). If the points $P_1,\dots,P_s$ are in linearly general position and $m_1 \ge m_2 \ge \cdots \ge m_s$, then $\mu \le \max\{m_1 + m_2 - 1, (m_1 + \cdots + m_s + n - 2)/n\}$ ([3]).

Then we will consider a problem of “ nice ” linkage over $\mathbb{F}_q$ (see [2] for general theory).

Theorem 3. *Fix integers $n \ge r \ge 2$ and a prime power q. Let $C \subset \mathbf{P}^n$ a smooth subscheme with pure codimension r defined over $\mathbb{F}_q$. Let μ be the first non-negative integer z such that $h^i(\mathbf{P}^n, \mathcal{I}_C(z-i)) = 0$ for all $i \ge 1$. Fix r integers $t_1 \ge \cdots \ge t_r \ge \mu + 1$. Assume $q \ge \sum_{i=1}^{r}(t_i^n - 1)t_i^{n^2}$. Then there are degree t_i hypersurfaces $A_i \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$ such that $A_1 \cap \cdots \cap A_r$ is a codimension r hypersurface containing C, reduced along C and smooth outside C*

In the statement of Theorem 3 we do not assume that C is connected or that it is geometrically connected. If C is not geometrically connected we do not assume that all the irreducible components of $C(\bar{\mathbb{F}}_q)$ are defined over $\mathbb{F}_q$.

2. THE PROOFS

Proof of Theorem 1. Since $\dim(Z) = 0$ we have $h^j(\mathbf{P}^n, \mathcal{I}_Z(t)) = 0$ for all $t \in \mathbb{Z}$ and all j such that either $j \ge 2$ and $t \ge -n$ or $2 \le j \le n-1$. Let μ be the first integer $t \ge -1$ such that $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) = 0$. Thus $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) = 0$ for all $t \ge \mu$ and $d \ge \mu + 1$. By Castelnuovo-Mumford's lemma the homogeneous ideal of Z is generated by forms of degree at most $\mu + 1$ and hence it is generated by forms of degree at most d. Let $v : M \to \mathbf{P}^n$ be the blowing-up of $\mathbf{P}^n$ at the points $P_1,\dots,P_s$. We have $R_*^j(\mathcal{O}_M) = 0$ for all $j \ge 1$ and $v_*(\mathcal{O}_M) = \mathcal{O}_{\mathbf{P}^n}$. Set $E_i := v^{-1}(P_i)$. Hence E_i, $1 \le i \le s$. Hence Pic$(M) \cong \mathbb{Z}^{\oplus s+1}$ and Pic(M) is freely generated by the classes of the line bundles $v^*(\mathcal{O}_{\mathbf{P}^n}(1))$ and $\mathcal{O}_M(E_i)$, $1 \le i \le s$. For all integers t, z, z_i. $1 \le i \le z$, set $\mathcal{L}_{t,z} := v^*(\mathcal{O}_{\mathbf{P}^n}(t)(-zE_1 - \cdots - zE_s))$ and $\mathcal{L}_{t,z_1,\dots,z_s} := v^*(\mathcal{O}_{\mathbf{P}^n}(t)(-z_1E_1 - z\cdots - z_sE_s))$. Since $P_i \in \mathbf{P}^n(\mathbb{F}_q)$ for all i, v, M, each E_i and all $\mathcal{L}_{t,z}$ and $\mathcal{L}_{t,z_1,\dots,z_s}$ are defined over $\mathbb{F}_q$. If $z_i \ge 0$ for all i, then $v_*(\mathcal{L}_{t,z_1,\dots z_s} = \mathcal{I}_{\cup_{i=1}^{s} z_iP_i}(t)$.

(a) Here we will check that $R_*^j(\mathcal{L}_{t,z_1,\dots,z_s}) = 0$ for all integers $j, t, z_1,\dots,z_s$ such that $j \ge 1$ and $z_i \ge 0$ for all i. By the projection formula it is sufficient to prove the case $t = 0$. The result is true if $z_i = 0$ for all i. Hence we may assume $z_i > 0$ for some i and use induction on the integer $z_1 + \cdots + z_s$. Hence we may assume that the result is true for the integers $z_1,\dots,z_{i-1}, z_i - 1, z_{i+1},\dots,z_s$. Set $B := \cup_{i=1}^{s} z_iE_i$

and $B' := B - E_i$. Thus we have the following exact sequence on M:

$$0 \to \mathcal{I}_B \to \mathcal{I}_{B'} \to \mathcal{O}_{E_i}(B') \to 0 \tag{1}$$

Apply the direct image functor to (1), the cohomology of $E_i \cong \mathbf{P}^{n-1}$ and that $\mathcal{O}_{E_i}(B')$ is a degree $z_i - 1$ line bundle on E_i.

(b) By part (a) and the definition of μ we have $h^j(M, \mathcal{L}_{t,m_1,\dots,m_s}) = 0$ and $h^0(M, \mathcal{L}_{t,m_1,\dots,m_s}) = \binom{n+t}{n} - \sum_{i=1}^{s} \binom{m_i+n-1}{n-1}$ for all $j \geq 1$, and $t \geq \mu$ and in particular for all $j \geq 1$ and $t \geq d-1$. In the same way we get that $h^1(M, \mathcal{L}_{t,z_1,\dots,z_s}(-E_i)) = 0$ for all $t \geq \mu + 1$.

(c) Here we will show that $\mathcal{L}_{t,m_1,\dots,m_s}$ is very ample for all $t \geq \mu + 1$ and in particular for $t = d$. It is sufficient to show the surjectivity of the restriction map $\rho_{A,t} : H^0(M, \mathcal{L}_{t,m_1,\dots,m_s}) \to H^0(A, \mathcal{L}_{t,m_1,\dots,m_s})$ for all zero-dimensional subschemes $A \subset M$ such that length$(A) = 2$. We distinguish six cases.

(i) A is reduced, say $A = \{Q, Q'\}$ with $Q \neq Q'$, and $A \cap (E_1 \cup \cdots \cup E_s) = \emptyset$;
(ii) A is not reduced and $Q := A_{red} \notin E_1 \cup \cdots \cup E_s$;
(iii) A is reduced, say $A = \{Q, Q'\}$ with $Q \neq Q'$, $Q \in E_i$, $Q \in E_j$ and $i \neq j$;
(iv) A is reduced, say $A = \{Q, Q'\}$ with $Q \neq Q'$, with $Q \in E_i$ and $Q' \notin E_1 \cup \cdots \cup E_s$;
(v) A is not reduced, $Q := A_{red} \in E_i$, and A is not contained in E_i;
(vi) $A \subset E_i$ for some i.

In cases (i), (ii), (iii), (iv), (v) the morphism $v|A : A \to \mathbf{P}^n$ is an embedding. In all these cases it is sufficient to use that the homogeneous ideal of Z is generated by forms of degree at most t. Now assume that we are in case (vi). We have $h^1(\mathbf{P}^n, \mathcal{I}_{Z'}(t-1)) = 0$ for all schemes $Z' \subset Z$. Take the set-up of part (a) with respect to the integers $z_j := m_j$ for all j. Apply the twist by $\mathcal{L}_{t,0,\dots,0}$ to the exact sequence (1), use the last vanishing of part (b) and that the line bundle $\mathcal{L}_{d,m_1,\dots,m_s}|E_i$ is the degree m_i line bundle on $E_i \cong \mathbf{P}^{n-1}$ and hence it is very ample.

(d) By part (c) the line bundle $\mathcal{L}_{d,m_1,\dots,m_s}$ is very ample. Notice that we have $\deg(\mathcal{L}_{d,m_1,\dots,m_s}) = d^n - \sum_{1=1}^{s} m_i^n = \delta$. By [1], Th. 1, there is a smooth $W \in |\mathcal{L}_{d,m_1,\dots,m_s}|$. Set $X := v(W)$. Now we consider the " Furthermore " part. We need to find W as above with the additional property that W is transversal to each E_i Since $\deg(\mathcal{L}_{d,m_1,\dots,m_s} \cap E_i) = m_i$, $\mathcal{L}_{d,m_1,\dots,m_s}$ embedds $E_i \cong \mathbf{P}^{n-1}$ by a subsystem of the degree m_i Verone embedding. Hence the embedded projective space has degree $m_i^{n-1} = \delta_i$. Thus its dual variety Δ_i in the projective space $|\mathcal{L}_{d,m_1,\dots,m_s}|$ has degree at most $(\delta_i)\delta_i^{n-1}$. The proof of [1], Lemma 1, and our assumption on q implies the existence of a hyperplane of $|\mathcal{L}_{d,m_1,\dots,m_s}|$ transversal to the image of M and to the images of all E_i. □

Proof of Theorem 2. We use the set-up introduced in the proof of Theorem 1. Now some of the line bundles $\mathcal{O}_M(E_i)$ may not be defined over $\mathbb{F}_q$, but v, M and all line bundles $\mathcal{L}_{t,z}$ are defined over $\mathbb{F}_q$. Furthermore for any $t \in \mathbb{Z}$ the line bundle $\mathcal{L}_{t,m_1,\dots,m_s}$ is defined over $\mathbb{F}_q$. Working over $\bar{F}_q$ the proof of Theorem 1 show that $\mathcal{L}_{d,m_1,\dots,m_s}$ is very ample. Hence we may again apply [1], Th. 1. □

Proof of Theorem 3. Let $v : M \to \mathbf{P}^n$ be the blowing-up of C. Since C is smooth, M is smooth. Set $E := v^{-1}(C)$. For all integers t set $\mathcal{L}_t := v^*((\mathcal{O}_{\mathbf{P}^n}(t)(-E)$. As in the proof of Theorem 1 it is easy to check that $\mathbb{L}_t$ is very ample for all $t \geq \mu = 1$. we again apply [1], Th. 1. Since a complete intersection in a smooth ambient has no embedded point, to check the existence of X which is reduced along C it is sufficient to test finitely many points of $C(\bar{\mathbb{F}}_q)$. □

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