Construction of singular hypersurfaces
and linkage over a finite field

E. Ballico

Dept. of Mathematics, University of Trento
38050 Povo (TN), Italy
ballico@science.unitn.it

Abstract. Here we prove two existence theorem over $\mathbb{F}_q$: existence of hypersurfaces with prescribed isolated singularities and existence of "smooth" linkage.

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1. The statements

Here we consider two existence theorems over $\mathbb{F}_q$. The corresponding constructions are obvious over $\mathbb{F}_q$ and the aim is just to find a relatively low prime power $q$ such that the same constructions may be done over $\mathbb{F}_q$. For any $P \in \mathbb{P}^n(\bar{\mathbb{F}}_q)$ and any integer $m > 0$ let $mP$ denote the infinitesimal neighborhood of order $m − 1$ of $P$ in $\mathbb{P}^n$. Set $0P = \emptyset$. In section 2 we will study the case of hypersurfaces with prescribed isolated singularities and prove the following result.

Theorem 1. Fix a prime power $q$, an integer $n \geq 2$, an integer $d > 0$, an integer $s$ such that $1 \leq s \leq (q^{n+1} - 1)/(q - 1)$, integers $m_i > 0$, and $s$ distinct points $P_1, \ldots, P_s \in \mathbb{P}^n(\mathbb{F}_q)$. Let $Z := \cup_{i=1}^s m_iP_i$ and assume $h^1(\mathbb{P}^n, \mathcal{I}_Z(d-1)) = 0$. Set $\delta := d^n − \sum_{i=1}^s m_i^n$ and $\delta_i := m_i^{n-1}$. Assume $q \geq (\delta-1)\delta^n$. Then there exists a degree $d$ hypersurface $X \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ and such that $\text{Sing}(X) \subseteq \{P_1, \ldots, P_s\}$. Assume $\text{Sing}(X) \subseteq \text{Sing}(\bar{\mathbb{F}}_q)$ if and only if $m_i \geq 2$, and $X$ has multiplicity $m_i$ at each $P_i$. Furthermore, if $q \geq (\delta-1)\delta^n + \sum_{i=1}^s (\delta_i-1)\delta_i^{n-1}$, then we may find $X$ such that $X$ has an ordinary multiple point with multiplicity $m_i$ at $P_i$, i.e. the tangent cone of $X$ at $P_i$ is a cone over a smooth degree $m_i$ hypersurface of $\mathbb{P}^{n-1}$.

When $P_1, \ldots, P_s \in \mathbb{P}^n(\bar{\mathbb{F}}_q)$, $P_i \notin \mathbb{P}^n(\mathbb{F}_q)$ for some $i$, but the set of all pairs $\{(P_1, m_1), \ldots, (P_s, m_s)\}$ is invariant for the natural action of the absolute Galois group of $\mathbb{F}_q$ we are able to prove the following result.

Theorem 2. Fix a prime power $q$, an integer $n \geq 2$, an integer $d > 0$, an integer $s$ such that $1 \leq s \leq (q^{n+1} - 1)/(q - 1)$, integers $m_i > 0$, and $s$ distinct points $P_1, \ldots, P_s \in \mathbb{P}^n(\bar{\mathbb{F}}_q)$. Let $Z := \cup_{i=1}^s \mathbb{P}^n(\mathbb{F}_q)$ and assume $h^1(\mathbb{P}^n, \mathcal{I}_Z(d-1)) = 0$. Assume that the scheme $Z$ and the inclusion of $Z$ in $\mathbb{P}^n$ are defined over $\mathbb{F}_q$, i.e. assume that the absolute Galois group of $\mathbb{F}_q$ acts trivially on the set of pairs

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\{(P_1, m_1), \ldots, (P_s, m_s)\}. Set \(\delta := d^n - \sum_{i=1}^s m_i^n\) and \(\delta_i := m_i^{n-1}\). Assume \(q \geq (\delta - 1)\delta^n\). Then there exists a degree \(d\) hypersurface \(X \subset \mathbb{P}^n\) defined over \(\mathbb{F}_q\) and such that \(\text{Sing}(X) \subseteq \{P_1, \ldots, P_s\}\), \(P_i \in \text{Sing}(X)\) if and only if \(m_i \geq 2\), and \(X\) has multiplicity \(m_i\) at each \(P_i\). Furthermore, if \(q \geq (\delta - 1)\delta^n + \sum_{i=1}^s (\delta_i - 1)\delta_i^{n-1}\), then we may find \(X\) such that \(X\) has an ordinary multiple point with multiplicity \(m_i\) at \(P_i\), i.e. the tangent cone of \(X\) at \(P_i\) is a cone over a smooth degree \(m_i\) hypersurface of \(\mathbb{P}^{n-1}\).

**Remark 1.** Take \(Z\) as in the statements of Theorems 1 and 2 and let \(\mu\) be the first integer \(t \geq -1\) such that \(h^i(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0\). Thus \(h^i(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0\) for all \(t \geq \mu\) and \(d \geq \mu + 1\). It is classical that \(\mu \leq m_1 + \cdots + m_s - 1\) and that we have equality if and only if the points \(P_1, \ldots, P_s\) are collinear (\([3]\)). If the points \(P_1, \ldots, P_s\) are in linearly general position and \(m_1 \geq m_2 \geq \cdots \geq m_s\), then \(\mu \leq \max\{m_1 + m_2 - 1, (m_1 + \cdots + m_s + n - 2)/n\}\) (\([3]\)).

Then we will consider a problem of “nice” linkage over \(\mathbb{F}_q\) (see [2] for general theory).

**Theorem 3.** Fix integers \(n \geq r \geq 2\) and a prime power \(q\). Let \(C \subset \mathbb{P}^n\) a smooth subscheme with pure codimension \(r\) defined over \(\mathbb{F}_q\). Let \(\mu\) be the first non-negative integer \(z\) such that \(h^i(\mathbb{P}^n, \mathcal{I}_C(z-i)) = 0\) for all \(i \geq 1\). Fix \(r\) integers \(t_1 \geq \cdots \geq t_r \geq \mu + 1\). Assume \(q \geq \sum_{i=1}^r (t_i^n - 1)t_i^2\). Then there are degree \(t_i\) hypersurfaces \(A_i \subset \mathbb{P}^n\) defined over \(\mathbb{F}_q\) such that \(A_1 \cap \cdots \cap A_r\) is a codimension \(r\) hypersurface containing \(C\), reduced along \(C\) and smooth outside \(C\).

In the statement of Theorem 3 we do not assume that \(C\) is connected or that it is geometrically connected. If \(C\) is not geometrically connected we do not assume that all the irreducible components of \(C(\mathbb{F}_q)\) are defined over \(\mathbb{F}_q\).

2. The proofs

**Proof of Theorem 1.** Since \(\dim(Z) = 0\) we have \(h^i(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0\) for all \(t \in \mathbb{Z}\) and all \(j\) such that either \(j \geq 2\) and \(t \geq -n\) or \(2 \leq j \leq n - 1\). Let \(\mu\) be the first integer \(t \geq -1\) such that \(h^i(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0\). Thus \(h^i(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0\) for all \(t \geq \mu\) and \(d \geq \mu + 1\). By Castelnuovo-Mumford’s lemma the homogeneous ideal of \(Z\) is generated by forms of degree at most \(\mu + 1\) and hence it is generated by forms of degree at most \(d\). Let \(v : M \to \mathbb{P}^n\) be the blowing-up of \(\mathbb{P}^n\) at the points \(P_1, \ldots, P_s\). We have \(R^s_j(\mathcal{O}_M) = 0\) for all \(j \geq 1\) and \(v_*(\mathcal{O}_M) = \mathcal{O}_{\mathbb{P}^n}\). Set \(E_i := v^{-1}(P_i)\). Hence \(E_i, 1 \leq i \leq s\). Hence \(\text{Pic}(M) \cong \mathbb{Z}^{\#s+1}\) and \(\text{Pic}(M)\) is freely generated by the classes of the line bundles \(v^*(\mathcal{O}_{\mathbb{P}^n}(1))\) and \(\mathcal{O}_M(E_i), 1 \leq i \leq s\). For all integers \(t, z, z_i, 1 \leq i \leq z\), set \(\mathcal{L}_{t, z} := v^*(\mathcal{O}_{\mathbb{P}^n}(t)(-zE_1 - \cdots - zE_s))\) and \(\mathcal{L}_{t, z_1, \ldots, z_s} := v^*(\mathcal{O}_{\mathbb{P}^n}(t)(-z_1E_1 - \cdots - z_sE_s))\). Since \(P_i \in \mathbb{P}^n(\mathbb{F}_q)\) for all \(i, v, M\), each \(E_i\) and all \(\mathcal{L}_{t, z}\) and \(\mathcal{L}_{t, z_1, \ldots, z_s}\) are defined over \(\mathbb{F}_q\). If \(z_i \geq 0\) for all \(i\), then \(v_*(\mathcal{L}_{t, z_1, \ldots, z_s}) = \mathcal{I}_{\mathcal{L}_{t, z_1, \ldots, z_s}}P_i(1)\).

(a) Here we will check that \(R^s_j(\mathcal{L}_{t, z_1, \ldots, z_s}) = 0\) for all integers \(j, t, z_1, \ldots, z_s\) such that \(j \geq 1\) and \(z_i \geq 0\) for all \(i\). By the projection formula it is sufficient to prove the case \(t = 0\). The result is true if \(z_i = 0\) for all \(i\). Hence we may assume \(z_i > 0\) for some \(i\) and use induction on the integer \(z_1 + \cdots + z_s\). Hence we may assume that the result is true for the integers \(z_1, \ldots, z_i-1, z_i - 1, z_{i+1}, \ldots, z_s\). Set \(B := \bigcup_{i=1}^s z_iE_i\)
and \( B' := B - E_i \). Thus we have the following exact sequence on \( M \):
\[
(1) \quad 0 \rightarrow \mathcal{I}_B \rightarrow \mathcal{I}_{B'} \rightarrow \mathcal{O}_{E_i}(B') \rightarrow 0
\]

Apply the direct image functor to (1), the cohomology of \( E_i \cong \mathbb{P}^{n-1} \) and that \( \mathcal{O}_{E_i}(B') \) is a degree \( z_i - 1 \) line bundle on \( E_i \).

(b) By part (a) and the definition of \( \mu \) we have \( h^j(M, \mathcal{L}_{t,1},...,m_s) = 0 \) and \( h^0(M, \mathcal{L}_{t,1},...,m_s) = (m^+_t)^{-1} \sum_i (m^+_t - 1) \) for all \( j \geq 1 \), and \( t \geq \mu \) and in particular for all \( j \geq 1 \) and \( t \geq d - 1 \). In the same way we get that \( h^1(M, \mathcal{L}_{t,1},...,m_s(-E_i)) = 0 \) for all \( t \geq \mu + 1 \).

(c) Here we will show that \( \mathcal{L}_{t,1},...,m_s \) is very ample for all \( t \geq \mu + 1 \) and in particular for \( t = d \). It is sufficient to show the surjectivity of the restriction map \( \rho_{A, t} : H^0(M, \mathcal{L}_{t,1},...,m_s) \rightarrow H^0(A, \mathcal{L}_{t,1},...,m_s) \) for all zero-dimensional subschemes \( A \subset M \) such that \( \text{length}(A) = 2 \). We distinguish six cases.

(i) \( A \) is reduced, say \( A = \{ Q, Q' \} \) with \( Q \neq Q' \), and \( A \cap (E_1 \cup \cdots \cup E_s) = \emptyset \);

(ii) \( A \) is not reduced and \( Q := A_{\text{red}} \notin E_1 \cup \cdots \cup E_s \);

(iii) \( A \) is reduced, say \( A = \{ Q, Q' \} \) with \( Q \neq Q' \), \( Q \in E_i \), \( Q \in E_j \) and \( i \neq j \);

(iv) \( A \) is reduced, say \( A = \{ Q, Q' \} \) with \( Q \neq Q' \), \( Q \in E_i \) and \( Q' \notin E_1 \cup \cdots \cup E_s \);

(v) \( A \) is not reduced, \( Q := A_{\text{red}} \in E_i \), and \( A \) is not contained in \( E_i \);

(vi) \( A \subset E_i \) for some \( i \).

In cases (i), (ii), (iii), (iv), (v) the morphism \( v|A : A \rightarrow \mathbb{P}^n \) is an embedding. In all these cases it is sufficient to use that the homogeneous ideal of \( Z \) is generated by forms of degree at most \( t \). Now assume that we are in case (vi). We have \( h^1(\mathbb{P}^n, \mathcal{I}_{Z}(t - 1)) = 0 \) for all schemes \( Z' \subset Z \). Take the set-up of part (a) with respect to the integers \( z_j := m_j \) for all \( j \). Apply the twist by \( \mathcal{L}_{t,0,...,0} \) to the exact sequence (1), use the last vanishing of part (b) and that the line bundle \( \mathcal{L}_{d,1},...,m_s}|E_i \) is the degree \( m_i \) line bundle on \( E_i \cong \mathbb{P}^{n-1} \) and hence it is very ample.

(d) By part (c) the line bundle \( \mathcal{L}_{d,1},...,m_s \) is very ample. Notice that we have \( \text{deg}(\mathcal{L}_{d,1},...,m_s) = d^n - \sum_{i=1}^{s} m_i^n = \delta \). By [1], Th. 1, there is a smooth \( W \in |\mathcal{L}_{d,1},...,m_s| \). Set \( X := v(W) \). Now we consider the “Furthermore” part. We need to find \( W \) as above with the additional property that \( W \) is transversal to each \( E_i \). Since \( \text{deg}(\mathcal{L}_{d,1},...,m_s \cap E_i) = m_i \), \( \mathcal{L}_{d,1},...,m_s \) embeds \( E_i \cong \mathbb{P}^{n-1} \) by a subsystem of the degree \( m_i \) Veronese embedding. Hence the embedded projective space has degree \( m_i^{n-1} = \delta \). Thus its dual variety \( \Delta_i \) in the projective space \( |\mathcal{L}_{d,1},...,m_s| \) has degree at most \( (\delta_i)^{n-1} \). The proof of [1], Lemma 1, and our assumption on \( q \) implies the existence of a hyperplane of \( |\mathcal{L}_{d,1},...,m_s| \) transversal to the image of \( M \) and to the images of all \( E_i \).

Proof of Theorem 2. We use the set-up introduced in the proof of Theorem 1. Now some of the line bundles \( \mathcal{O}_M(E_i) \) may not be defined over \( \overline{\mathbb{F}}_q \), but \( v, M \) and all line bundles \( \mathcal{L}_{t,z} \) are defined over \( \overline{\mathbb{F}}_q \). Furthermore for any \( t \in \mathbb{Z} \) the line bundle \( \mathcal{L}_{t,1},...,m_s \) is defined over \( \overline{\mathbb{F}}_q \). Working over \( \overline{\mathbb{F}}_q \) the proof of Theorem 1 show that \( \mathcal{L}_{d,1},...,m_s \) is very ample. Hence we may again apply [1], Th. 1.

Proof of Theorem 3. Let \( v : M \rightarrow \mathbb{P}^n \) be the blowing-up of \( C \). Since \( C \) is smooth, \( M \) is smooth. Set \( E := v^{-1}(C) \). For all integers \( t \) set \( \mathcal{L}_t := v^*((\mathcal{O}_{\mathbb{P}^n}(t)(-E)) \). As in the proof of Theorem 1 it is easy to check that \( \mathcal{L}_t \) is very ample for all \( t \geq \mu = 1 \). We again apply [1], Th. 1. Since a complete intersection in a smooth ambient has no embedded point, to check the existence of \( X \) which is reduced along \( C \) it is sufficient to test finitely many points of \( C(\overline{\mathbb{F}}_q) \).
References


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