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AN INTERESTING TABLE CONTAINED BINOMIAL COEFFICIENTS

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Abstract

In this study, we show that sum of the column elements of a table formed by a given recurrence relation are Fibonacci numbers and sum of the diagonal elements on the table are Padovan sequence.

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1 Introduction

Consider the infinite-dimensional matrix $A = (a_{i,j})$, where we define recursively each elements $a_{i,j}$ as follows and $i, j \geq 1$.

$$(1) \quad a_{i,j} = 1 \qquad if \quad i = j$$

$$(2) \quad a_{i,j} = 0 \qquad if \quad i > j \quad or \quad j \geq 2i + 1$$

$$(3) \quad a_{i,j} = a_{i-1,j-2} + a_{i-1,j-1} \qquad other \quad cases$$

Condition (1) implies that every element on the main diagonal is 1; condition (2) implies that every element below the main diagonal is zero, where $i > j$, and that every element above $a_{i,2i-1}$ and $a_{i,2i}$ elements is zero, where $j \geq 2i+1$. We can now employ condition (3) to compute the remaining elements of A: add the two consecutive elements $a_{i-1,j-2}$ and $a_{i-1,j-1}$. Using these straightforward observations, we can determine the various elements of A. Thus, we obtain the following table.

TABLE

i/j	1	2	3	4	5	6	7	8	9	10	11	12	13	
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	2	1	0	0	0	0	0	0	0	0	0	0
3	0	0	1	3	3	1	0	0	0	0	0	0	0	0
4	0	0	0	1	4	6	4	1	0	0	0	0	0	0
5	0	0	0	0	1	5	10	10	5	1	0	0	0	0
6	0	0	0	0	0	1	6	15	20	15	6	1	0	0
7	0	0	0	0	0	0	1	7	21	35	35	21	7	1
8	0	0	0	0	0	0	0	1	8	28	56	70	56	28
$\vdots$														

Clearly, this table is contained the Pascal-like triangle [1].

2 Results

The following results are obtained by the above table.

Corollary 2.1 *For $i, j \geq 1$*

$$\sum_{i=1}^{j} a_{i,j} = F_{j+1}.$$

Proof. *When $j = 1$,*

$$\sum_{i=1}^{1} a_{i,1} = a_{1,1} = 1 = F_{1+1}.$$

Now assume it is true for every positive integer $\leq k$:

$$\sum_{i=1}^{k} a_{i,k} = a_{1,k} + a_{2,k} + a_{3,k} + ... + a_{k,k} = F_{k+1}.$$

By the recurrence relation, we can write the following equations

$$a_{k+1,k+1} = a_{k,k-1} + a_{k,k}$$

$$a_{k,k+1} = a_{k-1,k-1} + a_{k-1,k}$$
$$\vdots$$
$$a_{3,k+1} = a_{2,k-1} + a_{2,k}$$
$$a_{2,k+1} = a_{1,k-1} + a_{1,k}$$
$$a_{1,k+1} = 0.$$

Adding these equations, we get

$$\sum_{i=1}^{k+1} a_{i,k+1} = \sum_{i=1}^{k} a_{i,k-1} + \sum_{i=1}^{k} a_{i,k}$$

and, $a_{k,k-1} = 0$ *by the condition* (2), *we obtain*

$$\sum_{i=1}^{k+1} a_{i,k+1} = \sum_{i=1}^{k-1} a_{i,k-1} + \sum_{i=1}^{k} a_{i,k}$$
$$= F_k + F_{k+1} = F_{k+2}.$$

Now we examine the sum of the diagonal elements on the table:

$$a_{1,1} = 1$$
$$a_{2,1} + a_{1,2} = 1$$
$$a_{3,1} + a_{2,2} + a_{1,3} = 1$$
$$\vdots$$
$$a_{n,1} + a_{n-1,2} + ... + a_{1,n} = K_n.$$

Therefore, we can take the sum of the n-th diagonal elements as follows:

$$K_n = \sum_{k=1}^{n} a_{n-k+1,k}.$$

$\{K_n\}$ sequence is a Padovan sequence[2].

Corollary 2.2 *$K_i + K_{i+1} = K_{i+3}$, where $i \geq 1$.*

Proof.

$$K_i + K_{i+1} = \sum_{k=1}^{i} a_{i-k+1,k} + \sum_{k=1}^{i+1} a_{i-k+2,k}$$

$$= (a_{i,1} + a_{i-1,2} + ... + a_{1,i}) + (a_{i+1,1} + a_{i,2} + ... + a_{1,i+1})$$

$$= (a_{i,1} + a_{i,2}) + (a_{i-1,2} + a_{i-1,3}) + ... + (a_{1,i} + a_{1,i+1}) + a_{i+1,1}$$

$$= a_{i+1,3} + a_{i,4} + ... + a_{2,i+2} + a_{i+1,1}.$$

Since $i + 1 > 1$, $a_{i+1,1} = 0$.
Similarly since $i + 3 > 1$, $i + 2 > 2$ and $i + 3 \geq 2.1 + 1$, $a_{i+3,1} = 0$, $a_{i+2,2} = 0$ and $a_{1,i+3} = 0$. Then

$$K_i + K_{i+1} = a_{i+3,1} + a_{i+2,2} + a_{i+1,3} + a_{i,4} + ... + a_{2,i+2} + a_{1,i+3}$$

$$= \sum_{k=1}^{i+1} a_{i-k+4,k} = K_{i+3}.$$

Corollary 2.3

$$K_{i+5} - K_{i+4} = K_i, \qquad i \geq 1.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers in Applications,* A Wiley-Interscience Publication, New York, 2001.

[2] www.research.att.com/ njas/sequences/index.html

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