The asymptotic formulas for the sum of squares of negative eigenvalues of the singular Sturm-Liouville operator

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Abstract

In this work, we find the asymptotic formulas for the sum of squares of negative eigenvalues of the operator $L$ which is formed by differential expression

$$
\ell(y) = -y''(x) - q(x)y(x)
$$

and with the boundary condition $y'(0) = 0$, in the space $L_2[0, \infty)$

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1 INTRODUCTION

Let us consider the differential expression

$$
\ell(y) = -y''(x) - q(x)y(x)
$$

(1.1)
in the space $L_2[0, \infty)$. Suppose that the function $q(x)$ which placed in these expressions satisfies the following conditions:

1.) $q(x)$ is continuous, monotonous decreasing and positive valued function in the interval $[0, \infty)$.
2.) $\lim_{x \to \infty} q(x) = 0$.

We denote the set of all functions that satisfy the following conditions in $L_2[0, \infty)$ by $D(L)$:

1.) $y'(x)$ is absolutely continuous in every finite interval $[a, b] \subset [0, \infty)$.
2.) $y'(0) = 0$.
3.) $\ell(y) = -y''(x) - q(x)y(x) \in L_2[0, \infty)$.

Let the operator $L$ be defined by $Ly = \ell(y)$ from $D(L)$ to $L_2[0, \infty)$.

$L : D(L) \to L_2[0, \infty)$ is self-adjoint operator.

Moreover, it is known that the operator $L$ is semi bounded below and negative part of its spectrum is discrete [5].

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ be negative eigenvalues of the operator $L$.

In this work, we have found some asymptotic formulas for the sum $\sum_{\lambda_j < -\epsilon} \lambda_j^2$ ($\epsilon > 0$), as $\epsilon \to +0$.

In the work [6], some asymptotic formulas have been found for number of negative eigenvalues of the operator $L$.

By assuming that the function $q(x)$ is satisfied some additional conditions, as in the work [2], an asymptotic formula of the form

\[
\sum_{\lambda_j < -\epsilon} \lambda_j = -(3\pi)^{-1}[1 + O(\epsilon^{t_0})] \int_{q(x) \geq \epsilon} [2q(x) + \epsilon] sqrtq(x) - \epsilon \, dx
\]

has been found for the sum $\sum_{\lambda_j < -\epsilon} \lambda_j$ of negative eigenvalues of the operator $L$, as $\epsilon \to 0$.

Here $t_0$ is a positive constant.

In the work [1], the asymptotic behaviour of the negative part of the spectrum of a differential operator with the operator coefficient has been investigated. In the work [3] the asymptotic formula for the number of eigenvalues of Sturm-Liouville operator with the operator coefficient which has singularity has been found.
2 SOME RELATIONS ABOUT EIGENVALUES.

Let $p$ be inverse function of the function $q$. We consider following conditions, where $\epsilon \in (0, q(0))$

a) Let $L_0$ and $L_1$ be operators in the space $L_2[0, p(\epsilon)]$ which are formed by expression (1.1) and with the boundary conditions

\[
y(0) = y(p(\epsilon)) = 0 \\
y'(0) = y'(p(\epsilon)) = 0
\]

respectively.

b) Let $L_{0i}$ and $L_{1i}$ be operators in the space $L_2[x_{i-1}, x_i])$ which are formed by expression (1.1) with boundary conditions

\[
y(x_{i-1}) = y(x_i) = 0 \quad (2.1) \\
y'(x_{i-1}) = y'(x_i)) = 0 \quad (2.2)
\]

respectively.

c) Let $\overline{T}_{0i}$ be an operator which formed by expression

\[
\ell(y) = -y''(x) - q(x_i)y(x)
\]

with boundary condition (2.1) in the space $L_2[x_{i-1}, x_i])$ and $\overline{T}_{1i}$ be an operator which formed by expression

\[
\ell(y) = -y''(x) - q(x_{i-1})y(x)
\]

with boundary condition (2.2) in the space $L_2[x_{i-1}, x_i])$.

Divide the interval $[0, p(0)]$ into the intervals at the length

\[
\delta = \frac{p(\epsilon)}{\| p^k(\epsilon) \| + 1} \quad (2.3)
\]

Here $k$ is a constant number which belongs to interval $(0,1)$ and $\epsilon$ is also a constant number which satisfies the conditions $\epsilon \in (0, q(0))$, $p^k(\epsilon) \geq 2$.

Let the partition points of the interval $[0, p(\epsilon)]$ be

\[
0 = x_0 < x_1 < \cdots < x_m = p(\epsilon)
\]

.
Let $n_{0i}(\alpha)$ and $\overline{n}_{0i}(\alpha)$ be numbers of eigenvalues smaller than $-\alpha (\alpha \in (0, \infty))$ of the operators $L_{0i}$ and $\overline{L}_{0i}$ respectively.

Instead of $n_{0i}(\epsilon)$ and $\overline{n}_{0i}(\epsilon)$ we will simply write $n_{0i}$ and $\overline{n}_{0i}$ respectively. Moreover, let $\mu_i(1) \leq \mu_i(2) \leq \mu_i(3) \leq \cdots$ be the eigenvalues of the operator $\overline{L}_{0i}$.

**Theorem 2.1** For the eigenvalues smaller than $-\epsilon$ of the operator $\overline{L}_{0i}$, the inequality

$$\sum_{m=1}^{\overline{n}_{0i}} \mu_i^2(m) > \frac{\delta}{15\pi} \sqrt{q(x_i) - \epsilon[8q^2(x_i) + 4q(x_i)\epsilon + 3\epsilon^2] - 2q^2(x_i)}$$

is satisfied.

**Proof:** Since the eigenvalues of the operator $\overline{L}_{0i}$ are of the form

$$\mu_i(m) = \left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 - q(x_i) \quad (m = 1, 2, \cdots)$$

then we have

$$\sum_{m=1}^{\overline{n}_{0i}} \mu_i^2(m) = \sum_{m=1}^{\overline{n}_{0i}} \left\{\left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 - q(x_i)\right\} = \sum_{m=1}^{\overline{n}_{0i}} \left\{\left(\frac{m\pi}{\delta}\right)^2 - q(x_i)\right\}$$

$$= \sum_{m=1}^{\overline{n}_{0i}} \left\{q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right\}^2$$

(2.4)

From the relation $(\frac{m\pi}{\delta})^2 \leq (\frac{t\pi}{\delta})^2 \quad (m \leq t \leq m + 1)$ we find

$$\left[q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right]^2 \geq \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 \quad (m \leq t \leq m + 1; 1 \leq m \leq \overline{n}_{0i} - 1)$$

Hence, we obtain

$$\int_{m}^{m+1} \left[q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right] dt > \int_{m}^{m+1} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right] dt$$

or

$$\left[q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right]^2 > \int_{m}^{m+1} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 dt \quad (1 \leq m \leq \overline{n}_{0i} - 1)$$
By using these inequalities and the relation (2.4) we find

\[ \sum_{m=1}^{n_0} \mu_i^2(m) = \sum_{m=1}^{n_0} \left[ q(x_i) - \left( \frac{m\pi}{\delta} \right)^2 \right] \geq \sum_{m=1}^{n_0-1} \left[ q(x_i) - \left( \frac{m\pi}{\delta} \right)^2 \right] \]

\[ > \sum_{m=1}^{n_0-1} \int_m^{m+1} \left[ q(x_i) - \left( \frac{t\pi}{\delta} \right)^2 \right] dt = \int_1^{n_0} \left[ q(x_i) - \left( \frac{t\pi}{\delta} \right)^2 \right] dt \]

\[ \geq \int_1^{n_0} \left[ q(x_i) - \left( \frac{t\pi}{\delta} \right)^2 \right] dt - q^2(x_i) \] (2.5)

Moreover, from the inequality \( \left( \frac{m\pi}{\delta} \right)^2 - q(x_i) < -\epsilon \) we obtain

\[ \frac{\delta}{\pi} \sqrt{q(x_i)} - \epsilon - 1 \leq n_0 \leq \frac{\delta}{\pi} \sqrt{q(x_i)} - \epsilon \] (2.6)

From (2.5) and (2.6), we find

\[ \sum_{m=1}^{n_0} \mu_i^2(m) > \int_0^{a-1} \left[ q(x_i) - \left( \frac{t\pi}{\delta} \right)^2 \right] dt - q^2(x_i) > \int_0^{a} \left[ q(x_i) - \left( \frac{t\pi}{\delta} \right)^2 \right] dt - 2q^2(x_i) \]

where \( a = \delta\pi^{-1} \sqrt{q(x_i)} - \epsilon. \)

Let us calculate the integral which is end of this expression.

\[ \int_0^a \left[ q(x_i) - \left( \frac{t\pi}{\delta} \right)^2 \right] dt = \int_0^a [q^2(x_i) - 2q(x_i)\pi^2\delta^{-2}t^2 + \pi^4\delta^{-4}t^4]dt \]

\[ = \left[ q^2(x_i)t - 2q(x_i)\pi^2\delta^{-2}t^3 + \pi^4\delta^{-4}t^5 \right]_0^a \]

\[ = q^2(x_i)a - 2q(x_i)\pi^2\delta^{-2}a^3 + \pi^4\delta^{-4}a^5 \]

\[ = q^2(x_i)\delta^{-1}\pi \sqrt{q(x_i)} - \epsilon - \frac{2}{3}q(x_i)\pi^2\delta^{-2}\pi^{-3}\delta^3[q(x_i) - \epsilon]^{3/2} \]

\[ + \frac{1}{5}\pi^4\delta^{-4}\pi^{-5}\delta^5[q(x_i) - \epsilon]^{5/2} \]

\[ = \frac{\delta}{\pi} \sqrt{q(x_i)} - \epsilon \left\{ q^2(x_i) - \frac{2}{3}q(x_i)[q(x_i) - \epsilon] + \frac{1}{5}[q(x_i) - \epsilon]^2 \right\} \]

\[ = \frac{\delta}{\pi} \sqrt{q(x_i)} - \epsilon \left\{ q^2(x_i) - \frac{2}{3}q^2(x_i) + \frac{2}{3}q^2(x_i)\epsilon + \frac{1}{5}q^2(x_i) \right\} \]

\[ - 2q(x_i)\epsilon + \frac{\epsilon^2}{5} \]
\[
= \frac{\delta}{\pi} \sqrt{q(x_i) - \epsilon} \left[ \frac{8}{15} q^2(x_i) + \frac{4}{15} q(x_i) \epsilon + \frac{\epsilon^2}{5} \right]
\]
\[
= \frac{\delta}{15\pi} \sqrt{q(x_i) - \epsilon} \left[ 8q^2(x_i) + 4q(x_i) \epsilon + 3\epsilon^2 \right] \quad (2.8)
\]

From (2.7) and (2.8) we obtain

\[
\sum_{m=1}^{\tilde{\pi}_i} \mu_i^2(m) > \frac{\delta}{15\pi} \sqrt{q(x_i) - \epsilon} \left[ 8q^2(x_i) + 4q(x_i) \epsilon + 3\epsilon^2 \right] - 2q^2(x_i). \quad \Box
\]

Let \( \gamma_{i1} \leq \gamma_{i2} \leq \cdots \) and \( \overline{\gamma}_{i1} \leq \overline{\gamma}_{i2} \leq \cdots \) be eigenvalues of the operators \( L_{1i} \) and \( \overline{L}_{1i} \) respectively and let us define \( n_{1i}(\alpha), \overline{n}_{1i}(\alpha), n_{1i}(\epsilon) \) and \( \overline{n}_{1i}(\epsilon) \) as follows:

\[
n_{1i}(\alpha) = \sum_{\gamma_{im} < -\alpha} 1, \quad \overline{n}_{1i}(\alpha) = \sum_{\overline{\gamma}_{im} < -\alpha} 1
\]

\[
n_{1i}(\epsilon) = n_{1i}, \quad \overline{n}_{1i}(\epsilon) = \overline{n}_{1i}.
\]

**Theorem 2.2** For the eigenvalues smaller than \( -\epsilon \) of the operator \( \overline{L}_{1i} \), the inequality

\[
\sum_{m=1}^{\overline{\pi}_i} (\overline{\gamma}_{im})^2 < \frac{\delta}{15\pi} f(x_{i-1}, \epsilon) + q^2(x_{i-1})
\]

is satisfied.

**Proof:** The eigenvalues of the operator \( \overline{L}_{1i} \) are of the form

\[
\overline{\gamma}_{im} = \left[ \frac{(m-1)\pi}{x_i - x_{i-1}} \right]^2 - q(x_{i-1}) \quad (m = 1, 2, \cdots ) \quad (2.9)
\]

By the relation \( \left[ \frac{(m-1)\pi}{x_i - x_{i-1}} \right]^2 \geq \left( \frac{t\pi}{\delta} \right)^2 \), \( (m-2 \leq t \leq m-1; m = 2, 3, \cdots ) \) we have

\[
\left\{ q(x_{i-1}) - \left[ \frac{(m-1)\pi}{\delta} \right]^2 \right\}^2 \leq \left\{ q(x_{i-1}) - \left( \frac{t\pi}{\delta} \right)^2 \right\} \quad (m-2 \leq t \leq m-1; 2 \leq m \leq \overline{\pi}_{1i})
\]

Hence, we have
\[
\int_{m-2}^{m-1} \left\{ q(x_{i-1}) - \left[ \frac{(m-1)\pi}{\delta} \right]^2 \right\}^2 dt < \int_{m-2}^{m-1} \left\{ q(x_{i-1}) - \left( \frac{t\pi}{\delta} \right)^2 \right\}^2 dt
\]

or

\[
\left\{ q(x_{i-1}) - \left[ \frac{(m-1)\pi}{\delta} \right]^2 \right\}^2 < \int_{m-2}^{m-1} \left\{ q(x_{i-1}) - \left( \frac{t\pi}{\delta} \right)^2 \right\}^2 dt \quad (2 \leq m \leq m_{1i})
\]

(2.10)

By using (2.9) and (2.10), we obtain

\[
\sum_{m=1}^{m_{1i}} (\gamma_{im})^2 = \sum_{m=1}^{m_{1i}} \left\{ q(x_{i-1}) - \left[ \frac{(m-1)\pi}{\delta} \right]^2 \right\}^2
\]

\[
= q^2(0) + \sum_{m=1}^{m_{1i}} \left\{ q(x_{i-1}) - \left[ \frac{(m-1)\pi}{\delta} \right]^2 \right\}^2
\]

\[
< q^2(x_{i-1}) + \sum_{m=2}^{m_{1i}} \int_{m-2}^{m-1} \left\{ q(x_{i-1}) - \left( \frac{t\pi}{\delta} \right)^2 \right\}^2 dt
\]

\[
= q^2(x_{i-1}) + \sum_{m=1}^{m_{1i}-1} \int_{0}^{m-1} \left[ q(x_{i-1}) - \left( \frac{t\pi}{\delta} \right) \right]^2 dt
\]

From the inequality \( \left[ \frac{(m-1)\pi}{\delta} \right]^2 - q(x_{i-1}) < -\epsilon \) we find

\[
\pi_{1i} < \frac{\delta}{\pi} \sqrt{q(x_{i-1}) - \epsilon} + 1
\]

we obtain

\[
\sum_{m=1}^{m_{1i}} (\gamma_{im})^2 < \int_{0}^{b} \left[ q(x_{i-1}) - \left( \frac{t\pi}{\delta} \right)^2 \right] dt + q^2(x_{i-1})
\]

(2.11)

Here, \( b = \frac{\delta}{\pi} \sqrt{q(x_{i-1}) - \epsilon} \) was taken

By using (2.8) and (2.11) we find

\[
\sum_{m=1}^{m_{1i}} (\gamma_{im})^2 < \frac{\delta}{15\pi} \sqrt{q(x_{i-1}) - \epsilon} \left[ 8q^2(x_{i-1}) + 4q(x_{i-1})\epsilon + 3\epsilon^2 \right] + q^2(x_{i-1})
\]

or
\[
\sum_{m=1}^{n} (\gamma_{im})^2 < \frac{\delta}{15\pi} f(x_{i-1}, \epsilon) + q^2(x_{i-1}). \Box
\]

Let \( N(\alpha), \ N_0(\alpha) \) and \( N_1(\alpha) \) be the numbers of the eigenvalues smaller than \(-\epsilon\) of the operators \( L, \ L_0 \) and \( L_1 \) respectively.

In the work [6], the inequalities

\[
N_0(\epsilon) \leq N(\epsilon) \leq N_1(\epsilon)
\]

are proved. By the similar way the inequalities

\[
N_0(\alpha) \leq N(\alpha) \leq N_1(\alpha)(\alpha \geq \epsilon)
\]  \tag{2.12}

can be proved.

Since \( q(x_i) \leq q(x) \leq q(x_{i-1}) \) in the interval \([x_{i-1}, \ x_i]\) then \( L_{0i} \leq \bar{T}_{0i} \) and \( L_{1i} \geq \bar{T}_{1i} \). In this case from [7], it is known that

\[
n_{0i}(\alpha) \geq \pi_{0i}(\alpha), \quad n_{1i}(\alpha) \leq \pi_{1i}(\alpha)
\]  \tag{2.13}

On the other hand, from the variation principles of R. Courant we have

\[
N_0(\alpha) \geq \sum_{i=1}^{M} n_{0i}(\alpha), \quad N_1(\alpha) = \sum_{i=1}^{M} n_{1i}(\alpha)
\]  \tag{2.14}

From [4], (2.12), (2.13) and (2.14) we find

\[
\sum_{i=1}^{M} \pi_{0i}(\alpha) \leq N(\alpha) \leq \sum_{i=2}^{M} \pi_{1i}(\alpha) + n_{11}(\alpha).
\]

By using the last relation the inequalities

\[
\sum_{i=1}^{M} \sum_{m=1}^{\mu^2_i(m)} \leq \sum_{j=1}^{N(\epsilon)} \lambda^2_j \leq \sum_{i=2}^{M} \sum_{m=1}^{\pi_{1i}(\alpha)} (\gamma_{im})^2 + \sum_{m=1}^{\pi_{1i}(\alpha)} \gamma_{1m}^2
\]  \tag{2.15}

can be proved.

**Theorem 2.3** If the function \( q \) satisfied the conditions 1) and 2) then for small positive values of \( \epsilon \), we have

\[
\sum_{j=1}^{N(\epsilon)} \lambda^2_j > \frac{1}{15\pi} \int_{\delta}^{p(\epsilon)} f(x, \epsilon)dx - cp^k(\epsilon)
\]  \tag{2.16}
Here, \( f(x, \epsilon) = \sqrt{q(x)} - \epsilon [8q^2(x) + 4q(x)\epsilon + 3\epsilon^2] \) and \( c \) is a positive constant.

**Proof:** Since the function \( q(x) \) supposed that monotonous decreasing then the function \( f(x, \epsilon) \) will be monotonous decreasing with respect to \( x \) for every \( \epsilon \) satisfying the conditions \( \epsilon \in (0, q(0)) \), \( p^k(\epsilon) \geq 2 \).

Therefore we have

\[
\delta f(x_i, \epsilon) = \int_{x_i}^{x_{i+1}} f(x_i, \epsilon) dx > \int_{x_i}^{x_{i+1}} f(x, \epsilon) dx \quad (1 \leq i \leq M - 1)
\]  

(2.18)

By using theorem 2.1 and (2.18), we find

\[
\sum_{i=1}^{x_{i+1}} \mu_i^2(m) > \frac{1}{15\pi} \int_{x_i}^{x_{i+1}} f(x, \epsilon) dx - 2q^2(0)
\]  

(2.19)

From (2.15) and (2.19), we obtain

\[
\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \sum_{m=1}^{n_{11}} \gamma^2 m + \frac{1}{15\pi} \int f(x, \epsilon) dx - \delta^{-1} p(\epsilon) q^2(0)
\]  

(2.17)

From (2.3) for small positive values of \( \epsilon \), we find

\[
M = \frac{p(\epsilon)}{\delta} = ||p^k(\epsilon)|| + 1 < 2p^k(\epsilon)
\]  

(2.21)

If we consider that \( x_1 = \delta \) and \( x_M = p(\epsilon) \) then from (2.20) and (2.21) we obtain

\[
\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int f(x, \epsilon) dx - cp^k(\epsilon)
\]

so the inequality (2.16) is proved. Now, let us prove that the inequality (2.17). Again, if we consider that for every \( \epsilon \) satisfying the conditions \( \epsilon \in (0, q(0)) \), \( p^k(\epsilon) \geq 2 \) the function \( f(x, \epsilon) \) is monotonous decreasing with respect to \( x \) then we obtain
\[ \delta f(x_{i-1}, \epsilon) = \int_{x_{i-2}}^{x_{i-1}} f(x_{i-1}, \epsilon) dx < \int_{x_{i-2}}^{x_{i-1}} f(x, \epsilon) dx \quad (2 \leq i \leq M) \]  

(2.22)

From Theorem 2.2 and the relation (2.22), we find

\[ \sum_{m=1}^{n_{11}} (\gamma_{im})^2 < \frac{1}{15\pi} \int_{x_{i-2}}^{x_{i-1}} f(x, \epsilon) dx + q^2(0) \]  

(2.23)

From (2.15) and (2.23), we have

\[ \sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \sum_{m=1}^{n_{11}} \gamma_{1m}^2 + \sum_{i=2}^{M} \left[ \frac{1}{15\pi} \int_{x_{i-2}}^{x_{i-1}} f(x, \epsilon) dx + q^2(0) \right] \]

\[ = \frac{1}{15\pi} \int_{0}^{x_{M-1}} f(x, \epsilon) dx + (M - 1)q^2(0) \]

\[ < \sum_{m=1}^{n_{11}} \gamma_{1m}^2 + \frac{1}{15\pi} \int_{0}^{x_{M}} f(x, \epsilon) dx + q(0) \]  

(2.24)

If we consider that \( x_M = p(\epsilon) \) and \( M = \frac{p(\epsilon)}{\delta} \) then from (2.24) we obtain

\[ \sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \sum_{m=1}^{n_{11}} \gamma_{1m}^2 + \frac{1}{15\pi} p(\epsilon) \int_{0}^{p(\epsilon)} f(x, \epsilon) dx + \delta^{-1}p(\epsilon)q^2(0). \square \]

For the sum \( \sum_{m=1}^{n_{11}} \gamma_{1m}^2 \) on the inequality (2.7), the inequality

\[ \sum_{m=1}^{n_{11}} \gamma_{1m}^2 < c_1 \int_{0}^{\delta} f(x, \epsilon) dx + c_1p^k(\epsilon) \]  

(2.25)

can be proved. From (2.3), (2.17) and (2.25), we obtain

\[ \sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \frac{1}{15\pi} p(\epsilon) \int_{0}^{p(\epsilon)} f(x, \epsilon) dx + c_2 \int_{0}^{\delta} f(x, \epsilon) dx + c_2p^k(\epsilon) \]  

(2.26)

Here, \( c_2 > 0 \) is a constant.
3 ASYMPTOTIC FORMULAS FOR THE SUM OF SQUARES OF THE NEGATIVE EIGENVALUES

In this section we will find some formulas for the sum \( \sum_{\lambda_j < -\epsilon} \lambda_j^2 \) as \( \epsilon \to +0 \).

First of all we suppose that the function \( q(x) \) satisfies following condition:

3) For every \( \eta > 0 \)

\[
\lim_{x \to \infty} q(x)x^{k_0-\eta} = \lim_{x \to \infty} [q(x)x^{k_0+\eta}]^{-1} = 0
\]

where \( k_0 \) is a constant which belongs to the interval \((0, \frac{2}{5})\).

Theorem 3.1 If the conditions 1) and 3) are satisfies then the asymptotic formula

\[
\sum_{\lambda_j < -\epsilon} \lambda_j^2 = (15\pi)^{-1}[1 + O(\epsilon^t)] \int_{q(x)\geq \epsilon} \sqrt{q(x) - \epsilon} \left[8q^2(x) + 4q(x)\epsilon + 3\epsilon^2\right] dx
\]

is satisfied when \( \epsilon \to +0 \). Here \( t_0 \) is a positive constant.

Proof: By Theorem 2.3, for small positive values of \( \epsilon \) we have

\[
\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_{0}^{p(\epsilon)} f(x, \epsilon) dx - \frac{1}{15\pi} \int_{0}^{\delta} f(x, \epsilon) dx - c p^k(\epsilon) \quad (3.1)
\]

For the proof, we will limit the expressions in second side of the inequality (3.1). Since \( f(x, \epsilon) > 0 \) and \( p(\epsilon) \) is monotonous decreasing, we have

\[
\int_{0}^{p(\epsilon)} f(x, \epsilon) dx > \int_{0}^{p(2\epsilon)} f(x, \epsilon) dx = \int_{0}^{p(2\epsilon)} \sqrt{q(x) - \epsilon} \left[8q^2(x) + 4q(x)\epsilon + 3\epsilon^2\right] dx
\]

\[
> \int_{0}^{p(2\epsilon)} \sqrt{q(x) - \epsilon(8\epsilon^2 + 4\epsilon^2 + 3\epsilon^2)} dx = 15\epsilon^2 \int_{0}^{p(2\epsilon)} \sqrt{q(x) - \epsilon} dx \quad (3.2)
\]

Since \( q(x) \geq q[p(2\epsilon)] = 2\epsilon \) in the interval \([0, p(2\epsilon)]\) from (3.2) we find

\[
\int_{0}^{p(\epsilon)} f(x, \epsilon) dx > \epsilon^{5/2} p(2\epsilon) \quad (3.3)
\]
If we consider that the function $q(x)$ satisfies the condition 3) and

$$\lim_{\epsilon \to \infty} p(\epsilon) = \infty$$

then we have

$$\lim_{\epsilon \to 0} [q(p(\epsilon))(p(\epsilon)^{k_0 + \eta})^{-1} = 0$$

Hence, for small positive values of $\epsilon$ we obtain

$$p(\epsilon) > \epsilon^{-\frac{1}{k_0 + \eta}} \quad (3.4)$$

From (3.3) and (3.4) we find

$$\int_0^{p(\epsilon)} f(x, \epsilon)dx > c_3 \epsilon^{\frac{5}{2} - \frac{1}{k_0 + \eta}} \quad (3.5)$$

Here $c_3$ is a positive constant. By using $q(x)$ satisfying the condition 3) we obtain

$$\int_0^\delta f(x, \epsilon)dx = \int_0^\delta \sqrt{q(x) - \epsilon[8q^2(x) + 4\epsilon q(x) + 3\epsilon^2]} dx
\leq \int_0^\delta \sqrt{q(x)}[8q^2(x) + 4\epsilon q(x) + 3\epsilon^2] dx
= 15\int_0^1 q^{5/2}(x)dx + 15\int_1^\delta q^{5/2}(x)dx < c_4 + c_4 \int_1^\delta \frac{q^{5/2}(x)}{x^{2(1-k)}} dx
< c_4 + c_4 \delta^{\frac{5(\eta-k_0)}{2}} \quad (3.6)$$

On the other hand, from (2.3) for small positive values of $\epsilon$ we find

$$\delta < p^{1-k}(\epsilon) \quad (k \in (0, 1)) \quad (3.7)$$

From (3.6) and (3.7) we find

$$\int_0^\delta f(x, \epsilon)dx < c_4 + c_4(p^{1-k}(\epsilon))^{\frac{5(\eta-k_0)+2}{2}} < c_5(p(\epsilon))^{\frac{5(\eta-k_0)+2(1-k)}{2}} \quad (3.8)$$

Again, by using the function $q(x)$ satisfying the condition 3), we obtain

$$\lim_{\epsilon \to 0} q(p(\epsilon))(p(\epsilon))^{k_0 - \eta} = 0.$$
Hence we find $\epsilon (p(\epsilon))^{k_0-\eta} < 1$ or

$$p(\epsilon) < \epsilon^{\frac{1}{k_0-\eta}}$$  \hspace{1cm} (3.9)

From (3.8) and (3.9) we obtain

$$\int_0^{\delta} f(x, \epsilon) dx < c_5 \epsilon^{\frac{-[5(\eta-k_0)+2(1-k)]}{2(k_0-\eta)}}$$  \hspace{1cm} (3.10)

Moreover from (3.9) we find

$$p^k(\epsilon) < \epsilon^{\frac{k}{k_0-\eta}}$$  \hspace{1cm} (3.11)

From (3.5), (3.10) and (3.11), we obtain

$$\frac{\int_0^{\delta} f(x, \epsilon) dx}{p(\epsilon)} < c_6 \epsilon^{\frac{-[5(\eta-k_0)+2(1-k)]}{2(k_0-\eta)}}$$  \hspace{1cm} (3.12)

$$\frac{\int_0^{\delta} f(x, \epsilon) dx}{p^k(\epsilon)} < c_6 \epsilon^{\frac{-k}{k_0-\eta}}$$  \hspace{1cm} (3.13)

Since $k_0 > 0$, the functions

$$\frac{-[5(\eta-k_0)+2(1-k)]}{2(k_0-\eta)} - \frac{5}{2} + \frac{1}{k_0+\eta} \quad \text{and} \quad \frac{k}{k_0+\eta} - \frac{5}{2} + \frac{1}{k_0+\eta}$$

are continuous with respect to $\eta$ at the point $\eta = 0$.

Consequently, for every $t > 0$, as $0 < \eta < \omega$ there is a number $\omega = \omega(t) > 0$ such that

$$\frac{-[5(\eta-k_0)+2(1-k)]}{2(k_0-\eta)} - \frac{5}{2} + \frac{1}{k_0+\eta} > \frac{k(2-5k_0)}{2k_0} - t$$  \hspace{1cm} (3.14)

$$\frac{k}{k_0-\eta} - \frac{5}{2} + \frac{1}{k_0+\eta} > \frac{-2-5k_0-2k}{2k_0} - t$$  \hspace{1cm} (3.15)

By now, we have considered $k$ as a constant which belongs to the interval $(0, 1)$.

Here if we take $k = \frac{2-5k_0}{4}$, $t = t_0 = \min \left\{ \frac{(2-5k_0)^2}{16}, \frac{2-5k_0}{8k_0} \right\}$ then from (3.14) and (3.15) we obtain
\[-\frac{5(\eta - k_0) + 2(1 - k)}{2(k_0 - \eta)} - \frac{5}{2} + \frac{1}{k_0 + \eta} > \frac{(2 - 5k_0)^2}{8k_0} - t_0\]

\[\geq \frac{(2 - 5k_0)^2}{8k_0} - \frac{(2 - 5k_0)^2}{16k_0} = \frac{(2 - 5k_0)^2}{16k_0} \geq t_0 \quad (3.16)\]

\[- \frac{k}{k_0 - \eta} - \frac{5}{2} + \frac{1}{k_0 + \eta} > \frac{2 - 5k_0}{4k_0} - t_0 \geq \frac{2 - 5k_0}{4k_0} - \frac{2 - 5k_0}{8k_0}\]

\[= \frac{2 - 5k_0}{8k_0} \geq t_0 \quad (3.17)\]

From (3.12), (3.13), (3.16) and (3.17) we find

\[\int_0^\delta f(x, \epsilon)dx \leq p(\epsilon) \int_0^\delta f(x, \epsilon)dx < c_6 t_0 \quad (3.18)\]

From (3.1), (3.18) and (3.19) we obtain

\[\sum_{j=1}^{N(\epsilon)} \lambda_j^2 \leq \frac{1}{15\pi} \int_0^\delta f(x, \epsilon)dx > 1 - c_7 t_0 \quad (3.20)\]

From (2.26), (3.18) and (3.19) we find

\[\sum_{j=1}^{N(\epsilon)} \lambda_j^2 \leq 1 + c_8 t_0 \quad (3.21)\]

From (3.20) and (3.21) we obtain the asymptotic formula

\[\frac{1}{15\pi} \int_{q(x) \geq \epsilon} f(x, \epsilon)dx - 1 = O(\epsilon^0)\]
or
\[
\sum_{-\lambda_j < -\epsilon} \lambda_j^2 = \frac{1}{15\pi} [1 + O(\epsilon^{0})] \int_{q(x) \geq \epsilon} \sqrt{q(x)} - \epsilon \left[8q^2(x) + 4\epsilon q(x) + 3\epsilon^2\right] dx
\]
as \epsilon \to 0.

Let us denote the functions of the form \(\ln_0 x = x, \ln_j x = \ln(\ln_{j-1} x)\) by \(\ln_j x\) \((j = 0, 1, 2 \cdots)\) and suppose that the function \(q(x)\) satisfies the following condition:

4.) There are a number \(\xi > 0\) and a natural number \(n\) so that the function \(q(x) - (\ln_n x)^{-\xi}\) is neither negative valued and nor monotonous increasing in an interval \([a, \infty)\) \((a > 0)\).

For large values of \(x\), we can prove the inequality
\[
\ln_n \left(\frac{x}{\ln x}\right) < \ln_n x - \ln^{1-n} x.
\]

Hence if the conditions 1), 2), 4) are satisfied then using the last inequality, for the small positive values of \(\epsilon\), the inequality
\[
q \left(\frac{p(\epsilon)}{\ln p(\epsilon)}\right) - \epsilon > (\ln p(\epsilon))^{-(\xi+1)(n+1)}
\]
(3.22)
can be proved.

**Theorem 3.2** If the function \(q(x)\) satisfies 1), 2) and 4) then the asymptotic formula
\[
\sum_{-\lambda_j < -\epsilon} \lambda_j^2 = (15\pi)^{-1} [1 + O(\epsilon^{-\beta})] \int_{q(x) \geq \epsilon} \sqrt{q(x)} - \epsilon \left[8q^2(x) + 4\epsilon q(x) + 3\epsilon^2\right] dx
\]
is satisfied as \(\epsilon \to 0\). Here \(\beta\) is a positive constant.

**Proof:** From the theorem 2.3 we obtain
\[
\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_{0}^{p(\epsilon)} f(x, \epsilon) dx - c_9 \delta - cp^k(\epsilon)
\]
(3.23)
for small positive values of \(\epsilon\).

From (2.3) and (3.23) we find
\[ N(\epsilon) \sum_{j=1}^{\lambda^2_j} > \frac{1}{15\pi} \int_0^{p(\epsilon)} f(x, \epsilon) dx - c_{10}p^{1/2}(\epsilon) \]  

(3.24)

where \( k = \frac{1}{2} \).

Here \( c_{10} \) is a positive constant. Let us restrict the integral \( \int_0^{p(\epsilon)} f(x, \epsilon) dx \) which is in the second side of the inequality (3.24).

We have

\[ \int_0^{p(\epsilon)} f(x, \epsilon) dx = \int_0^{p(\epsilon)} \sqrt{q(x)} - \epsilon \left[ 8q^2(x) + 4\epsilon q(x) + 3\epsilon^2 \right] dx > 15\epsilon^2 \int_0^{p(\epsilon)} \sqrt{q(x)} - \epsilon \]

(3.25)

For small positive values of \( \epsilon \) from (3.25) we find

\[ \int_0^{p(\epsilon)} f(x, \epsilon) dx > \epsilon^2 \int_0^{1/2f(\epsilon)} \sqrt{q(x)} - \epsilon dx > \frac{\epsilon^2 f(\epsilon)}{2} \sqrt{q(f(\epsilon))} - \epsilon \]

(3.26)

where \( f(\epsilon) = p(\epsilon) \ln^{-1} p(\epsilon) \).

From (3.22) and (3.26) we obtain

\[ \int_0^{p(\epsilon)} f(x, \epsilon) dx > \frac{\epsilon^2 p(\epsilon)}{2 \ln p(\epsilon)} \left( \ln p(\epsilon) \right)^{-1/2(\xi+1)(n+1)} > \epsilon^2 p^{1/2}(\epsilon) \]

\[ \frac{p^{1/2}(\epsilon)}{\int_0^{p(\epsilon)} f(x, \epsilon) dx} < \epsilon^{-2} p^{-1/4}(\epsilon) \]

(3.27)

Since the function \( q(x) \) satisfied the condition 4) we have

\[ \epsilon = q(p(\epsilon)) \geq (\ln p(\epsilon))^{-\xi} . \]

From here we find

\[ p(\epsilon) \geq e^{e^{-\frac{\xi}{4}}} \]

(3.28)

From (3.27) and (3.28) we obtain

\[ \frac{p^{1/2}(\epsilon)}{\int_0^{p(\epsilon)} f(x, \epsilon) dx} < \epsilon^{-2} e^{-\frac{1}{4} e^{-\xi}} < e^{-e^{-\beta}} \]

(3.29)
From (3.24) and (3.29) we find

\[
\sum_{-\lambda_j < -\epsilon} \frac{\lambda_j^2}{-\lambda_j < -\epsilon} \int_{\frac{1}{15\pi} q(x) \geq \epsilon} f(x, \epsilon) dx > 1 - c_{11} e^{-\epsilon - \beta}
\]  

(3.30)

From (2.26) we obtain

\[
\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \frac{1}{15\pi} \int_{0}^{p(\epsilon)} f(x, \epsilon) dx + c_{12} \delta + c_{2} p^k(\epsilon)
\]  

(3.31)

From (2.3) and (3.31) we have

\[
\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \frac{1}{15\pi} \int_{0}^{p(\epsilon)} f(x, \epsilon) dx + c_{13} p^{1/2}(\epsilon)
\]  

(3.32)

where again \( k = \frac{1}{2} \).

From (3.29) and (3.32) we obtain

\[
\sum_{-\lambda_j < -\epsilon} \frac{\lambda_j^2}{-\lambda_j < -\epsilon} \int_{\frac{1}{15\pi} q(x) \geq \epsilon} f(x, \epsilon) dx < 1 + c_{14} e^{-\epsilon - \beta}
\]  

(3.33)

From (3.30) and (3.33) we obtain the asymptotic formula

\[
\sum_{-\lambda_j < -\epsilon} \frac{\lambda_j^2}{-\lambda_j < -\epsilon} \int_{\frac{1}{15\pi} q(x) \geq \epsilon} f(x, \epsilon) dx = O\left(e^{-\epsilon - \beta}\right)
\]

or

\[
\sum_{-\lambda_j < -\epsilon} \lambda_j^2 = (15\pi)^{-1} \left[1 + O(e^{-\epsilon - \beta}) \right] \int_{\frac{1}{15\pi} q(x) \geq \epsilon} \sqrt{q(x) - \epsilon \left[8q^2(x) + 4\epsilon q(x) + 3\epsilon^2\right]} dx
\]

as \( \epsilon \to 0 \).

References


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