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The asymptotic formulas for the sum of squares of negative eigenvalues of the singular Sturm-Liouville operator

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Abstract

In this work, we find the asymptotic formulas for the sum of squares of negative eigenvalues of the operator L which is formed by differential expression

$$\ell(y) = -y''(x) - q(x)y(x)$$

and with the boundary condition $y'(0) = 0$, in the space $L_2[0, \infty)$

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1 INTRODUCTION

Let us consider the differential expression

$$\ell(y) = -y''(x) - q(x)y(x) \qquad (1.1)$$

in the space $L_2[0,\infty)$. Suppose that the function $q(x)$ which placed in these expressions satisfies the following conditions:

1.) $q(x)$ is continuous, monotonous decreasing and positive valued function in the interval $[0,\infty)$.

2.) $\lim\limits_{x\to\infty} q(x) = 0$.

We denote the set of all functions that satisfy the following conditions in $L_2[0,\infty)$ by $D(L)$:

1.) $y'(x)$ is absolutely continuous in every finite interval $[a,b] \subset [0,\infty)$.

2.) $y'(0) = 0$.

3.) $\ell(y) = -y''(x) - q(x)y(x) \in L_2[0,\infty)$.

Let the operator L be defined by $Ly = \ell(y)$ from $D(L)$ to $L_2[0,\infty)$.

$L : D(L) \to L_2[0,\infty)$ is self-adjoint operator.

Moreover, it is known that the operator L is semi bounded below and negative part of its spectrum is discrete [5].

Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ be negative eigenvalues of the operator L. In this work, we have found some asymptotic formulas for the sum $\sum\limits_{\lambda_j<-\epsilon} \lambda_j^2$ $(\epsilon > 0)$, as $\epsilon \to +0$.

In the work [6], some asymptotic formulas have been found for number of negative eigenvalues of the operator L.

By assuming that the function $q(x)$ is satisfied some additional conditions, as in the work [2], an asymptotic formula of the form

$$\sum_{\lambda_j<-\epsilon} \lambda_j = -(3\pi)^{-1}[1 + O(\epsilon^{t_0})] \int\limits_{q(x)\ge\epsilon} [2q(x) + \epsilon]\, sqrtq(x) - \epsilon \; dx$$

has been found for the sum $\sum\limits_{\lambda_j<-\epsilon} \lambda_j$ of negative eigenvalues of the operator L, as $\epsilon \to 0$.

Here t_0 is a positive constant.

In the work [1] , the asymptotic behaviour of the negative part of the spectrum of a differential operator with the operator coefficient has been investigated. In the work [3] the asymptotic formula for the number of eigenvalues of Sturm-Liouville operator with the operator coefficient which has singularity has been found.

2 SOME RELATIONS ABOUT EIGENVALUES.

Let p be inverse function of the function q. We consider following conditions, where $\epsilon \in (0, q(0))$

a) Let L_0 and L_1 be operators in the space $L_2[0, p(\epsilon)])$ which are formed by expression (1.1) and with the boundary conditions

$$y(0) = y(p(\epsilon)) = 0$$

$$y'(0) = y'(p(\epsilon)) = 0$$

respectively.

b) Let L_{0i} and L_{1i} be operators in the space $L_2[x_{i-1}, x_i])$ which are formed by expression (1.1) with boundary conditions

$$y(x_{i-1}) = y(x_i) = 0 \tag{2.1}$$

$$y'(x_{i-1}) = y'(x_i)) = 0 \tag{2.2}$$

respectively.

c) Let $\overline{L}_{0i}$ be an operator which formed by expression

$$\ell(y) = -y''(x) - q(x_i)y(x)$$

with boundary condition (2.1) in the space $L_2[x_{i-1}, x_i])$ and $\overline{L}_{1i}$ be an operator which formed by expression

$$\ell(y) = -y''(x) - q(x_{i-1})y(x)$$

with boundary condition (2.2) in the space $L_2[x_{i-1}, x_i])$.

Divide the interval $[0, p(0)]$ into the intervals at the length

$$\delta = \frac{p(\epsilon)}{[|\ p^k(\epsilon)\ |] + 1} \tag{2.3}$$

Here k is a constant number which belongs to interval $(0, 1)$ and ϵ is also a constant number which satisfies the conditions $\epsilon \in (0, q(0))$, $p^k(\epsilon) \geq 2$.
Let the partition points of the interval $[0, p(\epsilon)]$ be

$$0 = x_0 < x_1 < \cdots < x_m = p(\epsilon)$$

.

Let $n_{0i}(\alpha)$ and $\overline{n}_{0i}(\alpha)$ be numbers of eigenvalues smaller than $-\alpha(\alpha \in (0,\infty))$ of the operators L_{0i} and $\overline{L}_{0i}$ respectively.
Instead of $n_{0i}(\epsilon)$ and $\overline{n}_{0i}(\epsilon)$ we will simply write n_{0i} and $\overline{n}_{0i}$ respectively. Moreover, let $\mu_i(1) \leq \mu_i(2) \leq \mu_i(3) \leq \cdots$ be the eigenvalues of the operator $\overline{L}_{0i}$.

Theorem 2.1 *For the eigenvalues smaller than* $-\epsilon$ *of the operator* $\overline{L}_{0i}$ *, the inequality*

$$\sum_{m=1}^{\overline{n}_{0i}} \mu_i^2(m) > \frac{\delta}{15\pi}\sqrt{q(x_i-\epsilon}[8q^2(x_i)+4q(x_i)\epsilon+3\epsilon^2]-2q^2(x_i)$$

is satisfied.

Proof: Since the eigenvalues of the operator $\overline{L}_{0i}$ are of the form

$$\mu_i(m) = \left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 - q(x_i) \qquad (m = 1, 2, \cdots)$$

then we have

$$\sum_{m=1}^{\overline{n}_{0i}} \mu_i^2(m) = \sum_{m=1}^{\overline{n}_{0i}} \left\{\left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 - q(x_i)\right\}^2 = \sum_{m=1}^{\overline{n}_{0i}} \left\{\left(\frac{m\pi}{\delta}\right)^2 - q(x_i)\right\}^2$$

$$= \sum_{m=1}^{\overline{n}_{0i}} \left\{q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right\}^2 \tag{2.4}$$

From the relation $(\frac{m\pi}{\delta})^2 \leq (\frac{t\pi}{\delta})^2 \quad (m \leq t \leq m+1)$ we find

$$\left[q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right]^2 \geq \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 \quad (m \leq t \leq m+1\,;\ 1 \leq m \leq \overline{n}_{0i} - 1)$$

Hence, we obtain

$$\int_m^{m+1} \left[q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right] dt > \int_m^{m+1} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right] dt$$

or

$$\left[q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right]^2 > \int_m^{m+1} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 dt \quad (1 \leq m \leq \overline{n}_{0i} - 1)$$

By using these inequalities and the relation (2.4) we find

$$\begin{aligned}\sum_{m=1}^{\overline{n}_{0i}} \mu_i^2(m) &= \sum_{m=1}^{\overline{n}_{0i}} \left[q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right]^2 \geq \sum_{m=1}^{\overline{n}_{0i}-1} \left[q(x_i) - \left(\frac{m\pi}{\delta}\right)^2\right]^2 \\ &> \sum_{m=1}^{\overline{n}_{0i}-1} \int_m^{m+1} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 dt = \int_1^{n_{0i}} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 dt \\ &\geq \int_1^{n_{0i}} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 dt - q^2(x_i) \qquad (2.5)\end{aligned}$$

Moreover, from the inequality $(\frac{m\pi}{\delta})^2 - q(x_i) < -\epsilon$ we obtain

$$\frac{\delta}{\pi}\sqrt{q(x_i)-\epsilon} - 1 \leq \overline{n}_{0i} < \frac{\delta}{\pi}\sqrt{q(x_i)-\epsilon} \qquad (2.6)$$

From (2.5) and (2.6), we find

$$\sum_{m=1}^{\overline{n}_{0i}} \mu_i^2(m) > \int_0^{a-1} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 dt - q^2(x_i) > \int_0^{a} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 dt - 2q^2(x_i) \qquad (2.7)$$

where $a = \delta\pi^{-1}\sqrt{q(x_i)-\epsilon}$.

Let us calculate the integral which is end of this expression.

$$\begin{aligned}\int_0^{a} \left[q(x_i) - \left(\frac{t\pi}{\delta}\right)^2\right]^2 dt &= \int_0^{a} [q^2(x_i) - 2q(x_i)\pi^2\delta^{-2}t^2 + \pi^4\delta^{-4}t^4]dt \\ &= [q^2(x_i)t - 2q(x_i)\pi^2\delta^{-2}\frac{t^3}{3} + \pi^4\delta^{-4}\frac{t^5}{5}]\Big|_0^a \\ &= q^2(x_i)a - 2q(x_i)\pi^2\delta^{-2}\frac{a^3}{3} + \pi^4\delta^{-4}\frac{a^5}{5} \\ &= q^2(x_i)\delta\pi^{-1}\sqrt{q(x_i)-\epsilon} - \frac{2}{3}q(x_i)\pi^2\delta^{-2}\pi^{-3}\delta^3[q(x_i)-\epsilon]^{3/2} \\ &\quad + \frac{1}{5}\pi^4\delta^{-4}\pi^{-5}\delta^5[q(x_i)-\epsilon]^{5/2} \\ &= \frac{\delta}{\pi}\sqrt{q(x_i)-\epsilon}\left\{q^2(x_i) - \frac{2}{3}q(x_i)[q(x_i)-\epsilon] + \frac{1}{5}[q(x_i)-\epsilon]^2\right\} \\ &= \frac{\delta}{\pi}\sqrt{q(x_i)-\epsilon}\left\{q^2(x_i) - \frac{2}{3}q^2(x_i) + \frac{2}{3}q(x_i)\epsilon + \frac{1}{5}q^2(x_i) \right. \\ &\quad \left. - \frac{2}{5}q(x_i)\epsilon + \frac{\epsilon^2}{5}\right\}\end{aligned}$$

$$= \frac{\delta}{\pi}\sqrt{q(x_i)-\epsilon}\left[\frac{8}{15}q^2(x_i)+\frac{4}{15}q(x_i)\epsilon+\frac{\epsilon^2}{5}\right]$$

$$= \frac{\delta}{15\pi}\sqrt{q(x_i)-\epsilon}\left[8q^2(x_i)+4q(x_i)\epsilon+3\epsilon^2\right] \qquad (2.8)$$

From (2.7)and (2.8) we obtain

$$\sum_{m=1}^{\overline{n}_{0i}}\mu_i^2(m) > \frac{\delta}{15\pi}\sqrt{q(x_i)-\epsilon}\left[8q^2(x_i)+4q(x_i)\epsilon+3\epsilon^2\right]-2q^2(x_i).\square$$

Let $\gamma_{i1}\le\gamma_{i2}\le\cdots$ and $\overline{\gamma}_{i1}\le\overline{\gamma}_{i2}\le\cdots$ be eigenvalues of the operators L_{1i} and $\overline{L}_{1i}$ respectively and let us define $n_{1i}(\alpha)$, $\overline{n}_{1i}(\alpha)$, $n_{1i}(\epsilon)$ and $\overline{n}_{1i}(\alpha)$ as follows:

$$n_{1i}(\alpha)=\sum_{\gamma_{im}<-\alpha}1\ , \qquad \overline{n}_{1i}(\alpha)=\sum_{\overline{\gamma}_{im}<-\alpha}1$$

$$n_{1i}(\epsilon)=n_{1i}\ , \qquad \overline{n}_{1i}(\epsilon)=\overline{n}_{1i}.$$

Theorem 2.2 *For the eigenvalues smaller than* $-\epsilon$ *of the operator* $\overline{L}_{1i}$, *the inequality*

$$\sum_{m=1}^{\overline{n}_{1i}}(\overline{\gamma}_{im})^2 < \frac{\delta}{15\pi}f(x_{i-1},\epsilon)+q^2(x_{i-1})$$

is satisfied.

Proof: The eigenvalues of the operator $\overline{L}_{1i}$ are of the form

$$\overline{\gamma}_{im}=\left[\frac{(m-1)\pi}{x_i-x_{i-1}}\right]^2-q(x_{i-1}) \qquad (m=1,2,\cdots) \qquad (2.9)$$

By the relation $\left[\frac{(m-1)\pi}{x_i-x_{i-1}}\right]^2 \ge \left(\frac{t\pi}{\delta}\right)^2$, $(m-2\le t\le m-1;\ m=2,3,\cdots)$ we have

$$\left\{q(x_{i-1})-\left[\frac{(m-1)\pi}{\delta}\right]^2\right\}^2 \le \left[q(x_{i-1})-\left(\frac{t\pi}{\delta}\right)^2\right] \quad (m-2\le t\le m-1; 2\le m\le \overline{n}_{1i})$$

Hence, we have

$$\int_{m-2}^{m-1}\left\{q(x_{i-1})-\left[\frac{(m-1)\pi}{\delta}\right]^2\right\}^2 dt<\int_{m-2}^{m-1}\left\{q(x_{i-1})-\left(\frac{t\pi}{\delta}\right)^2\right\}^2 dt$$

or

$$\left\{q(x_{i-1})-\left[\frac{(m-1)\pi}{\delta}\right]^2\right\}^2<\int_{m-2}^{m-1}\left\{q(x_{i-1})-\left(\frac{t\pi}{\delta}\right)^2\right\}^2 dt \qquad (2\le m\le \overline{n}_{1i}) \tag{2.10}$$

By using (2.9) and (2.10), we obtain

$$\sum_{m=1}^{\overline{n}_{1i}}(\overline{\gamma}_{im})^2=\sum_{m=1}^{\overline{n}_{1i}}\left\{q(x_{i-1})-\left[\frac{(m-1)\pi}{\delta}\right]^2\right\}^2$$

$$=q^2(0)+\sum_{m=2}^{\overline{n}_{1i}}\left\{q(x_{i-1})-\left[\frac{(m-1)\pi}{\delta}\right]^2\right\}^2$$

$$<q^2(x_{i-1})+\sum_{m=2}^{\overline{n}_{1i}}\int_{m-2}^{m-1}\left\{q(x_{i-1})-\left(\frac{t\pi}{\delta}\right)^2\right\}^2 dt$$

$$=q^2(x_{i-1})+\int_{0}^{\overline{n}_{1i}-1}\left[q(x_{i-1})-\left(\frac{t\pi}{\delta}\right)^2\right]^2 dt$$

From the inequality $\left[\frac{(m-1)\pi}{\delta}\right]^2-q(x_{i-1})<-\epsilon$ we find

$$\overline{n}_{1i}<\frac{\delta}{\pi}\sqrt{q(x_{i-1})-\epsilon}+1$$

we obtain

$$\sum_{m=1}^{\overline{n}_{1i}}(\overline{\gamma}_{im})^2<\int_0^b\left[q(x_{i-1})-\left(\frac{t\pi}{\delta}\right)^2\right]dt+q^2(x_{i-1}) \tag{2.11}$$

Here, $b=\frac{\delta}{\pi}\sqrt{q(x_{i-1})-\epsilon}$ was taken

By using (2.8) and (2.11) we find

$$\sum_{m=1}^{\overline{n}_{1i}}(\overline{\gamma}_{im})^2<\frac{\delta}{15\pi}\sqrt{q(x_{i-1})-\epsilon}[8q^2(x_{i-1})+4q(x_{i-1})\epsilon+3\epsilon^2]+q^2(x_{i-1})$$

or

$$\sum_{m=1}^{\overline{n}_{1i}} (\overline{\gamma}_{im})^2 < \frac{\delta}{15\pi} f(x_{i-1}, \epsilon) + q^2(x_{i-1}). \Box$$

Let $N(\alpha)$, $N_0(\alpha)$ and $N_1(\alpha)$ be the numbers of the eigenvalues smaller than $-\epsilon$ of the operators L, L_0 and L_1 respectively.

In the work [6], the inequalities

$$N_0(\epsilon) \leq N(\epsilon) \leq N_1(\epsilon)$$

are proved. By the similar way the inequalities

$$N_0(\alpha) \leq N(\alpha) \leq N_1(\alpha)(\alpha \geq \epsilon) \tag{2.12}$$

can be proved.

Since $q(x_i) \leq q(x) \leq q(x_{i-1})$ in the interval $[x_{i-1},\ x_i]$ then $L_{0i} \leq \overline{L}_{0i}$ and $L_{1i} \geq \overline{L}_{1i}$. In this case from [7], it is known that

$$n_{0i}(\alpha) \geq \overline{n}_{0i}(\alpha), \quad n_{1i}(\alpha) \leq \overline{n}_{1i}(\alpha) \tag{2.13}$$

On the other hand, from the variation principles of R. Courant we have

$$N_0(\alpha) \geq \sum_{i=1}^{M} n_{0i}(\alpha), \quad N_1(\alpha) = \sum_{i=1}^{M} n_{1i}(\alpha) \tag{2.14}$$

From [4], (2.12), (2.13) and (2.14) we find

$$\sum_{i=1}^{M} \overline{n}_{0i}(\alpha) \leq N(\alpha) \leq \sum_{i=2}^{M} \overline{n}_{1i}(\alpha) + n_{11}(\alpha).$$

By using the last relation the inequalities

$$\sum_{i=1}^{M} \sum_{m=1}^{\overline{n}_{0i}} \mu_i^2(m) \leq \sum_{j=1}^{N(\epsilon)} \lambda_j^2 \leq \sum_{i=2}^{M} \sum_{m=1}^{\overline{n}_{1i}} (\overline{\gamma}_{im})^2 + \sum_{m=1}^{n_{11}} \gamma_{1m}^2 \tag{2.15}$$

can be proved.

Theorem 2.3 *If the functionq satisfied the conditions 1) and 2) then for small positive values of* ϵ*, we have*

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_{\delta}^{p(\epsilon)} f(x,\ \epsilon)dx - cp^k(\epsilon) \tag{2.16}$$

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \sum_{m=1}^{n_{11}} \gamma_{1m}^2 + \frac{1}{15\pi} \int_{\delta}^{p(\epsilon)} f(x,\,\epsilon)dx - \delta^{-1}p(\epsilon)q^2(0) \qquad (2.17).$$

Here, $f(x,\,\epsilon) = \sqrt{q(x)-\epsilon}[8q^2(x) + 4q(x)\epsilon + 3\epsilon^2]$ *and* c *is a positive constant.*

Proof : Since the function $q(x)$ supposed that monotonous decreasing then the function $f(x,\epsilon)$ will be monotonous decreasing with respect to x for every ϵ satisfying the conditions $\epsilon \in (0,\ q(0))$, $p^k(\epsilon) \geq 2$.

Therefore we have

$$\delta f(x_i,\,\epsilon) = \int_{x_i}^{x_{i+1}} f(x_i,\,\epsilon)dx > \int_{x_i}^{x_{i+1}} f(x,\,\epsilon)dx \quad (1 \leq i \leq M-1) \qquad (2.18)$$

By using theorem 2.1 and (2.18), we find

$$\sum_{i=1}^{\overline{n}_{0i}} \mu_i^2(m) > \frac{1}{15\pi} \int_{x_i}^{x_{i+1}} f(x,\,\epsilon)dx - 2q^2(0) \qquad (2.19)$$

From (2.15) and (2.19), we obtain

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \sum_{i=1}^{M-1} \left[\frac{1}{15\pi} \int_{x_i}^{x_{i+1}} f(x,\,\epsilon)dx - 2q^2(0) \right] > \frac{1}{15\pi} \int_{x_1}^{x_M} f(x,\,\epsilon)dx - 2q^2(0)M \qquad (2.20)$$

From (2.3) for small positive values of ϵ, we find

$$M = \frac{p(\epsilon)}{\delta} = [|p^k(\epsilon)|] + 1 < 2p^k(\epsilon) \qquad (2.21).$$

If we consider that $x_1 = \delta$ and $x_M = p(\epsilon)$ then from (2.20) and (2.21) we obtain

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_{\delta}^{p(\epsilon)} f(x,\,\epsilon)dx - cp^k(\epsilon)$$

so the inequality (2.16) is proved. Now, let us prove that the inequality (2.17). Again, if we consider that for every ϵ satisfying the conditions $\epsilon \in (0,\ q(0))$, $p^k(\epsilon) \geq 2$ the function $f(x,\epsilon)$ is monotonous decreasing with respect to x then we obtain

$$\delta f(x_{i-1},\, \epsilon) = \int\limits_{x_{i-2}}^{x_{i-1}} f(x_{i-1},\, \epsilon)dx < \int\limits_{x_{i-2}}^{x_{i-1}} f(x,\, \epsilon)dx \quad (2 \leq i \leq M) \tag{2.22}$$

From Theorem 2.2 and the relation (2.22), we find

$$\sum_{m=1}^{\overline{n}_{1i}} (\overline{\gamma}_{im})^2 < \frac{1}{15\pi} \int\limits_{x_{i-2}}^{x_{i-1}} f(x,\, \epsilon)dx + q^2(0) \tag{2.23}$$

From (2.15) and (2.23), we have

$$\begin{aligned} \sum_{j=1}^{N(\epsilon)} \lambda_j^2 &< \sum_{m=1}^{n_{11}} \gamma_{1m}^2 + \sum_{i=2}^{M} \left[\frac{1}{15\pi} \int\limits_{x_{i-2}}^{x_{i-1}} f(x,\, \epsilon)dx + q^2(0) \right] \\ &= \frac{1}{15\pi} \int\limits_{0}^{x_{M-1}} f(x,\, \epsilon)dx + (M-1)q^2(0) \\ &< \sum_{m=1}^{n_{11}} \gamma_{1m}^2 + \frac{1}{15\pi} \int\limits_{0}^{x_M} f(x,\, \epsilon)dx + q(0) \end{aligned} \tag{2.24}$$

If we consider that $x_M = p(\epsilon)$ and $M = \frac{p(\epsilon)}{\delta}$ then from (2.24) we obtain

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \sum_{m=1}^{n_{11}} \gamma_{1m}^2 + \frac{1}{15\pi} \int\limits_{0}^{p(\epsilon)} f(x,\, \epsilon)dx + \delta^{-1} p(\epsilon) q^2(0). \square$$

For the sum $\sum\limits_{m=1}^{n_{11}} \gamma_{1m}^2$ on the inequality (2.7), the inequality

$$\sum_{m=1}^{n_{11}} \gamma_{1m}^2 < c_1 \int\limits_{0}^{\delta} f(x,\, \epsilon)dx + c_1 p^k(\epsilon) \tag{2.25}$$

can be proved. From (2.3), (2.17) and (2.25), we obtain

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \frac{1}{15\pi} \int\limits_{0}^{p(\epsilon)} f(x,\, \epsilon)dx + c_2 \int\limits_{0}^{\delta} f(x,\, \epsilon)dx + c_2 p^k(\epsilon) \tag{2.26}$$

Here, $c_2 > 0$ is a constant.

3 ASYMPTOTIC FORMULAS FOR THE SUM OF SQUARES OF THE NEGATIVE EIGENVALUES

In this section we will find some formulas for the sum $\sum\limits_{\lambda_j<-\epsilon} \lambda_j^2$ as $\epsilon \to +0$. First of all we suppose that the function $q(x)$ satisfies following condition:

3) For every $\eta > 0$

$$\lim_{x\to\infty} q(x)x^{k_0-\eta} = \lim_{x\to\infty} [q(x)x^{k_0+\eta}]^{-1} = 0$$

where k_0 is a constant which belongs to the interval $(0,\ \frac{2}{5})$.

Theorem 3.1 *If the conditions 1) and 3) are satisfies then the asymptotic formula*

$$\sum_{-\lambda_j<-\epsilon} \lambda_j^2 = (15\pi)^{-1}[1+O(\epsilon^{t_0})] \int\limits_{q(x)\geq\epsilon} \sqrt{q(x)-\epsilon}\Big[8q^2(x)+4q(x)\epsilon+3\epsilon^2\Big]dx$$

is satisfied when $\epsilon \to +0$. *Here* t_0 *is a positive constant.*

Proof: By Theorem 2.3, for small positive values of ϵ we have

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi}\int\limits_0^{p(\epsilon)} f(x,\ \epsilon)dx - \frac{1}{15\pi}\int\limits_0^{\delta} f(x,\ \epsilon)dx - cp^k(\epsilon) \tag{3.1}$$

For the proof, we will limit the expressions in second side of the inequality (3.1). Since $f(x,\epsilon) > 0$ and $p(\epsilon)$ is monotonous decreasing, we have

$$\int\limits_0^{p(\epsilon)} f(x,\ \epsilon)dx > \int\limits_0^{p(2\epsilon)} f(x,\ \epsilon)dx = \int\limits_0^{p(2\epsilon)} \sqrt{q(x)-\epsilon}\Big[8q^2(x)+4q(x)\epsilon+3\epsilon^2\Big]dx$$

$$> \int\limits_0^{p(2\epsilon)} \sqrt{q(x)-\epsilon}(8\epsilon^2+4\epsilon^2+3\epsilon^2)dx = 15\epsilon^2 \int\limits_0^{p(2\epsilon)} \sqrt{q(x)-\epsilon}dx \tag{3.2}$$

Since $q(x) \geq q[p(2\epsilon)] = 2\epsilon$ in the interval $[0, p(2\epsilon)]$ from (3.2) we find

$$\int\limits_0^{p(\epsilon)} f(x,\ \epsilon)dx > \epsilon^{5/2}p(2\epsilon) \tag{3.3}$$

If we consider that the function $q(x)$ satisfies the condition 3) and

$$\lim_{\epsilon\to\infty} p(\epsilon) = \infty$$

then we have

$$\lim_{\epsilon\to 0}[q(p(\epsilon))(p(\epsilon))^{k_0+\eta}]^{-1} = 0$$

Hence, for small positive values of ϵ we obtain

$$p(\epsilon) > \epsilon^{-\frac{1}{k_0+\eta}} \tag{3.4}$$

From (3.3) and (3.4) we find

$$\int_0^{p(\epsilon)} f(x,\, \epsilon)dx > c_3\epsilon^{\frac{5}{2}-\frac{1}{k_0+\eta}} \tag{3.5}$$

Here c_3 is a positive constant. By using $q(x)$ satisfying the condition 3) we obtain

$$\begin{aligned}\int_0^{\delta} f(x,\, \epsilon)dx &= \int_0^{\delta} \sqrt{q(x)-\epsilon}\Big[8q^2(x)+4\epsilon q(x)+3\epsilon^2\Big]dx \\ &< \int_0^{\delta} \sqrt{q(x)}\Big[8q^2(x)+4\epsilon q(x)+3\epsilon^2\Big]dx \\ &= 15\int_0^1 q^{5/2}(x)dx + 15\int_1^{\delta} q^{5/2}(x)dx < c_4 + c_4\int_1^{\delta} x^{\frac{5(\eta-k_0)}{2}}dx \\ &< c_4 + c_4\delta^{\frac{5(\eta-k_0)}{2}+1} \end{aligned}\tag{3.6}$$

On the other hand, from (2.3) for small positive values of ϵ we find

$$\delta < p^{1-k}(\epsilon) \qquad (k\in(0,\, 1)) \tag{3.7}$$

From (3.6) and (3.7) we find

$$\int_0^{\delta} f(x,\, \epsilon)dx < c_4 + c_4(p^{1-k}(\epsilon))^{\frac{5(\eta-k_0)+2}{2}} < c_5(p(\epsilon))^{\frac{[5(\eta-k_0)+2](1-k)}{2}} \tag{3.8}$$

Again, by using the function $q(x)$ satisfying the condition 3), we obtain

$$\lim_{\epsilon\to 0} q(p(\epsilon))(p(\epsilon))^{k_0-\eta} = 0.$$

Hence we find $\epsilon(p(\epsilon))^{k_0-\eta} < 1$ or

$$p(\epsilon) < \epsilon^{-\frac{1}{k_0-\eta}} \tag{3.9}$$

From(3.8) and (3.9) we obtain

$$\int_0^{\delta} f(x,\ \epsilon)dx < c_5\epsilon^{\frac{-[5(\eta-k_0)+2](1-k)}{2(k_0-\eta)}} \tag{3.10}$$

Moreover from (3.9) we find

$$p^k(\epsilon) < \epsilon^{\frac{k}{k_0-\eta}} \tag{3.11}$$

From (3.5), (3.10) and (3.11), we obtain

$$\frac{\int_0^{\delta} f(x,\ \epsilon)dx}{\int_0^{p(\epsilon)} f(x,\ \epsilon)dx} < c_6\epsilon^{\frac{-[5(\eta-k_0)+2](1-k)}{2(k_0-\eta)}-\frac{5}{2}+\frac{1}{k_0+\eta}} \tag{3.12}$$

$$\frac{p^k(\epsilon)}{\int_0^{p(\epsilon)} f(x,\ \epsilon)dx} < c_6\epsilon^{-\frac{k}{k_0+\eta}-\frac{5}{2}+\frac{1}{k_0+\eta}} \tag{3.13}$$

Since $k_0 > 0$, the functions

$$\frac{-[5(\eta-k_0)+2](1-k)}{2(k_0-\eta)} - \frac{5}{2} + \frac{1}{k_0+\eta} \quad \text{and} \quad -\frac{k}{k_0+\eta} - \frac{5}{2} + \frac{1}{k_0+\eta}$$

are continuous with respect to η at the point $\eta = 0$.

Consequently, for every $t > 0$, as $0 < \eta < \omega$ there is a number $\omega = \omega(t) > 0$ such that

$$\frac{-[5(\eta-k_0)+2](1-k)}{2(k_0-\eta)} - \frac{5}{2} + \frac{1}{k_0+\eta} > -\frac{k(2-5k_0)}{2k_0} - t \tag{3.14}$$

$$-\frac{k}{k_0-\eta} - \frac{5}{2} + \frac{1}{k_0+\eta} > -\frac{2-5k_0-2k}{2k_0} - t \tag{3.15}$$

By now, we have considered k as a constant which belongs to the interval $(0,\ 1)$.

Here if we take $k = \frac{2-5k_0}{4}$, $t = t_0 = \min\left\{\frac{(2-5k_0)^2}{16},\ \frac{2-5k_0}{8k_0}\right\}$ then from (3.14) and (3.15) we obtain

$$\frac{-[5(\eta-k_0)+2](1-k)}{2(k_0-\eta)}-\frac{5}{2}+\frac{1}{k_0+\eta}>\frac{(2-5k_0)^2}{8k_0}-t_0$$

$$\geq \frac{(2-5k_0)^2}{8k_0}-\frac{(2-5k_0)^2}{16k_0}=\frac{(2-5k_0)^2}{16k_0}\geq t_0 \tag{3.16}$$

$$-\frac{k}{k_0-\eta}-\frac{5}{2}+\frac{1}{k_0+\eta}>\frac{2-5k_0}{4k_0}-t_0\geq\frac{2-5k_0}{4k_0}-\frac{2-5k_0}{8k_0}$$

$$=\frac{2-5k_0}{8k_0}\geq t_0 \tag{3.17}$$

From (3.12), (3.13), (3.16) and (3.17) we find

$$\frac{\int_0^{\delta} f(x,\,\epsilon)dx}{\int_0^{p(\epsilon)} f(x,\,\epsilon)dx}<c_6\epsilon^{t_0} \tag{3.18}$$

$$\frac{\int_0^{\delta} f(x,\,\epsilon)dx}{\int_0^{p(\epsilon)} f(x,\,\epsilon)dx}<c_6 t_0 \tag{3.19}$$

From (3.1), (3.18) and (3.19) we obtain

$$\frac{\sum_{j=1}^{N(\epsilon)}\lambda_j^2}{\frac{1}{15\pi}\int_0^{p(\epsilon)} f(x,\,\epsilon)dx}>1-c_7\epsilon^{t_0} \tag{3.20}$$

From (2.26), (3.18) and (3.19) we find

$$\frac{\sum_{j=1}^{N(\epsilon)}\lambda_j^2}{\frac{1}{15\pi}\int_0^{p(\epsilon)} f(x,\,\epsilon)dx}<1+c_8\epsilon^{t_0} \tag{3.21}$$

From (3.20) and (3.21) we obtain the asymptotic formula

$$\frac{\sum_{j=1}^{N(\epsilon)}\lambda_j^2}{\frac{1}{15\pi}\int_{q(x)\geq\epsilon} f(x,\,\epsilon)dx}-1=O(\epsilon^{t_0})$$

or

$$\sum_{-\lambda_j<-\epsilon} \lambda_j^2 = \frac{1}{15\pi}[1+O(\epsilon^{t_0})] \int\limits_{q(x)\geq\epsilon} \sqrt{q(x)-\epsilon}\Big[8q^2(x)+4\epsilon q(x)+3\epsilon^2\Big]dx$$

as $\epsilon \to 0$.

Let us denote the functions of the form $\ln_0 x = x,\ \ln_j x = \ln(\ln_{j-1} x)$ by $ln_j x \quad (j = 0,\ 1,\ 2\cdots)$ and suppose that the function $q(x)$ satisfies the following condition:

4.) There are a number $\xi > 0$ and a natural number n so that the function $q(x) - (\ln_n x)^{-\xi}$ is neither negative valued and nor monotonous increasing in an interval $[a,\ \infty)\ (a > 0)$.

For large values of x, we can prove the inequality

$$\ln_n\left(\frac{x}{\ln x}\right) < \ln_n x - \ln^{1-n} x.$$

Hence if the conditions 1), 2), 4) are satisfied then using the last inequality, for the small positive values of ϵ, the inequality

$$q\left(\frac{p(\epsilon)}{\ln p(\epsilon)}\right) - \epsilon > (\ln p(\epsilon))^{-(\xi+1)(n+1)} \tag{3.22}$$

can be proved.

Theorem 3.2 *If the function $q(x)$ satisfies 1), 2) and 4) then the asymptotic formula*

$$\sum_{-\lambda_j<-\epsilon} \lambda_j^2 = (15\pi)^{-1}[1+O(e^{-\epsilon^{-\beta}})] \int\limits_{q(x)\geq\epsilon} \sqrt{q(x)-\epsilon}\Big[8q^2(x)+4\epsilon q(x)+3\epsilon^2\Big]dx$$

is satisfied as $\epsilon \to 0$. Here β is a positive constant.

Proof: From the theorem 2.3 we obtain

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int\limits_0^{p(\epsilon)} f(x,\ \epsilon)dx - c_9\delta - cp^k(\epsilon) \tag{3.23}$$

for small positive values of ϵ.

From (2.3) and (3.23) we find

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_0^{p(\epsilon)} f(x,\ \epsilon)dx - c_{10} p^{1/2}(\epsilon) \tag{3.24}$$

where $k = \frac{1}{2}$.

Here c_{10} is a positive constant. Let us restrict the integral $\int_0^{p(\epsilon)} f(x,\epsilon)dx$ which is in the second side of the inequality (3.24).

We have

$$\int_0^{p(\epsilon)} f(x,\ \epsilon)dx = \int_0^{p(\epsilon)} \sqrt{q(x)-\epsilon}\Big[8q^2(x)+4\epsilon q(x)+3\epsilon^2\Big]dx > 15\epsilon^2 \int_0^{p(\epsilon)} \sqrt{q(x)-\epsilon} \tag{3.25}$$

For small positive values of ϵ from (3.25) we find

$$\int_0^{p(\epsilon)} f(x,\ \epsilon)dx > \epsilon^2 \int_{1/2f(\epsilon)}^{f(\epsilon)} \sqrt{q(x)-\epsilon}dx > \frac{\epsilon^2 f(\epsilon)}{2}\sqrt{q(f(\epsilon))-\epsilon} \tag{3.26}$$

where $f(\epsilon) = p(\epsilon)\ln^{-1} p(\epsilon)$.

From (3.22) and (3.26) we obtain

$$\int_0^{p(\epsilon)} f(x,\ \epsilon)dx > \frac{\epsilon^2 p(\epsilon)}{2\ln p(\epsilon)}\Big(\ln p(\epsilon)\Big)^{-1/2(\xi+1)(n+1)} > \epsilon^2 p^{1/2}(\epsilon)$$

$$\frac{p^{1/2}(\epsilon)}{\int_0^{p(\epsilon)} f(x,\ \epsilon)dx} < \epsilon^{-2} p^{-1/4}(\epsilon) \tag{3.27}$$

Since the function $q(x)$ satisfied the condition 4) we have

$$\epsilon = q(p(\epsilon)) \geq (\ln_n p(\epsilon))^{-\xi}.$$

From here we find

$$p(\epsilon) \geq e^{\epsilon^{-\frac{1}{\xi}}} \tag{3.28}$$

From (3.27) and (3.28) we obtain

$$\frac{p^{1/2}(\epsilon)}{\int_0^{p(\epsilon)} f(x,\ \epsilon)dx} < \epsilon^{-2} e^{-\frac{1}{4}\epsilon^{-\frac{1}{\xi}}} < e^{-\epsilon^{-\beta}} \tag{3.29}$$

From (3.24) and (3.29) we find

$$\frac{\sum\limits_{-\lambda_j<-\epsilon}\lambda_j^2}{\frac{1}{15\pi}\int\limits_{q(x)\geq\epsilon}f(x,\ \epsilon)dx} > 1-c_{11}e^{-\epsilon^{-\beta}} \tag{3.30}$$

From (2.26) we obtain

$$\sum_{j=1}^{N(\epsilon)}\lambda_j^2 < \frac{1}{15\pi}\int\limits_0^{p(\epsilon)} f(x,\ \epsilon)dx + c_{12}\delta + c_2p^k(\epsilon) \tag{3.31}$$

From (2.3) and (3.31) we have

$$\sum_{j=1}^{N(\epsilon)}\lambda_j^2 < \frac{1}{15\pi}\int\limits_0^{p(\epsilon)} f(x,\ \epsilon)dx + c_{13}p^{1/2}(\epsilon) \tag{3.32}$$

where again $k=\frac{1}{2}$.

From (3.29) and (3.32) we obtain

$$\frac{\sum\limits_{-\lambda_j<-\epsilon}\lambda_j^2}{\frac{1}{15\pi}\int\limits_{q(x)\geq\epsilon}f(x,\ \epsilon)dx} < 1+c_{14}e^{-\epsilon^{-\beta}} \tag{3.33}$$

From (3.30) and (3.33) we obtain the asymptotic formula

$$\frac{\sum\limits_{-\lambda_j<-\epsilon}\lambda_j^2}{\frac{1}{15\pi}\int\limits_{q(x)\geq\epsilon}f(x,\ \epsilon)dx} - 1 = O\left(e^{-\epsilon^{-\beta}}\right)$$

or

$$\sum_{-\lambda_j<-\epsilon}\lambda_j^2 = (15\pi)^{-1}[1+O(e^{-\epsilon^{-\beta}})]\int\limits_{q(x)\geq\epsilon}\sqrt{q(x)-\epsilon}\Big[8q^2(x)+4\epsilon q(x)+3\epsilon^2\Big]dx$$

as $\epsilon\to 0$.

References

[1] E.E. Adigüzelov, M. Bairamoglu and F.G. Maksudov, "On asymtoties of spectrum and trace high order differential operator coefficients", Doga-Turkish journal of mathematics, vol. 17, No:2, 1993.

[2] E.E. Adigüzelov, Z. Oer, "Asymptotic Expansion for The Sum of Negative Eigenvalues of Sturm-Liouville Operator Given in Semi-Axis", Journal of YTU, 26-35, 2000/1.

[3] E.E. Adigüzelov, Ö. Bakși and A. Bayramoglu, "The asymptotic behaviour of the negative part of the spectrum of Sturm-Liouville opertor with the operator coefficient which has singularity", International Journal of Differential Equations and Applications, Vol. 6, No. 3, 315-329, 2002.

[4] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol 1, New York, 1970.

[5] M.A. Naimark, Linear Differential Operators, part I, II, London, 1968.

[6] B.Y. Skaçek, Asymptod of Negative Part of Spectrum of One Dimensioned Differential Operators, Pribl. metodi reșeniya differn. uraveniy, Kiev, 1963.

[7] V.I. Smirnov, A Course of Higher Mathematics, Vol. 5, New York Pergamon Press, 1964.

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