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On the reduced free product and cohomology of topological groups with non-abelian coefficient

H. Sahleh

Department of Mathematics
Guilan University
P.O. Box 1914, Rasht, Iran
sahleh@guilan.ac.ir

Abstract

Let G_1 and G_2 be topological groups. In this paper we define the reduced free product of G_1 ,G_2 .We use singular complex to define a cohomology with non-abelian coefficient and an exact sequence.

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Introduction

All spaces are assumed to be completely regular and Hausdorff.The free topological group is in Makov sense[6]. For a space X, $S(X)$ means the singular complex on X. In section 1 ,we define the cohomology with non-abelian coefficient in dimensions 0 and 1. In section 2, the reduced free product is introduced. By this notion we find an exact sequence which involves the cohomology in dimensions 0 and 1.

1 Non-abelian cohomology

In [1] the cohomology group $H^*(G, A)$ where A is a trivial G-module is defined. In this section we generalize it for the non-abelian group A and will define the non-abelian cohomology for a topological group in dimensions 0 and 1.

Let G be a topological group and $A \subseteq G$ a subspace ,Π a group(not necessarily abelian).We write the group operation additively.

A 1-cochain c^1 is a continuous map which assigns to each $u \in S(G)$ an element $c^1(u_i) \in \Pi$ such that $c^1(u_i) = 0$ for $u_i \in S(A)$. The set of these elements is denoted by $C^1(G, A; \Pi)$. Similarly for a 0-cochain $e^0 \in C^0(G, A; \Pi)$. A 1-cocycle z^1 is an element of $C^1(X, A; \Pi)$ such that for any 2-simplex u_2

$$z^1(u_2^{(0)}) + z^1(u_2^{(1)}) + z^1(u_2^{(2)}) = 0$$

where $u_2^{(i)}$ is the ith face of u_2. We denote the set of 1-cocycles by $Z^1(X, A)$. For a 0-cocycle z^0 ,for any u_i , $z^0(u_1^{(0)}) = z^0(u_1^{(1)})$. Two 1-cocycles z^1 and z'^2 are cohomologous if there is a 0-cochain $c^0 \in C^0(X, A)$ such that

$$z'^1(u_1) = -c^0(u_1^{(1)}) + z^1(u_1) + c^0(u_1^{(0)})$$

for all u_1. This is an equivalence relation [8] . We denote the set of equivalent classes by

$$H^1(X, A)$$

The zero of this set is the class containing the trivial cycle which assigns 0 to every u_i.

For dimension 0 we define $H^0(X, A) = Z^0(X, A)$. Two homomorphisms are in the same class if they differ by an inner automorphism . We denote the set of homomorphic-class by $Homcl(-, -)$.

Topological fundamental group

Let X be a metrizable space and $x_0 \in X$, $C_{x_0}(X) = \{f : [0, 1] \to X, \textit{suchthat } f \textit{ is continuous and } f(0) = f(1) = x_0\}$. Endow $C_{x_0}(X)$ with the topology of uniform convergence.

Definition 1.1 The topological fundamental group $\pi_1(X, x_0)$ is the set of path components of $C_{x_0}(X)$ endowed with the quotient topology under canonical surjection $q : C_{x_0}(X) \to \pi_1(X, p)$ satisfying $q(f) = q(g)$ if and only if f and g belong to the same path component of $C_{x_0}(X)$. Thus $U \subset \pi_1(X, x_0)$ is open if and only if $q^{-1}(U)$ is open in $C_{x_0}(X)$.

The topological fundamental group is a topological group[2,proposition 1]. For more on this subject see [2].

Lemma 1.2 *Let X be path connected ,metrizable and $x_0 \in X$ then $H^1(X, x_0)$ may be identified with the set $Hom(\pi_1(X, x_0), \Pi)$ and $H^1(X, \Pi)$ with $Homcl(\pi_1(X, x_0), \Pi)$.*

Proof. If z^1 is a 1-cocycle of X over Π we associate with it a homomorphism $\theta : \pi_1(X, x_0) \to \Pi$ as follows: For any $\gamma \in \pi_1(X, x_0)$ and any sequence of paths $u_1^1, u_1^2, ..., u_1^j$ forming a closed path representing γ we have $\theta(\gamma) = z^1(u_1^1) + z^1(u_1^2) + ... + z^1(u_1^j)$ This yields a unique correspondences

$$H^1(X, x_0; \Pi) \to Hom(\pi_1(X, x_0), \Pi)$$

$$H^1(X, \Pi) \to Homcl(\pi_1(X, x_0), \Pi)$$

which are both 1-1 and onto and the zero elements of the left are corresponding to the trivial homomorphism (or homomorphism-class) on the right.

By "homomorphism" of cohomology groups we mean it maps one set into the other and carries zero to zero. An"isomorphism" must be 1-1 and carries zero into zero. A map $f : (X, A) \to (Y, B)$ induces a homomorphism $f^* : H^1(Y, B; \Pi) \to H^1(X, A; \Pi)$ in the usual way. Given a triple (X, A, B), we define a map

$$\delta : H^0(A, B; \pi) \to H^1(X, A; \pi) \tag{1.3}$$

as follows: let $h^0 \in H^0(A, B; \Pi)$ we define $c^0 \in C^0(X, B; \Pi)$ by $c^0(u_0) = h^0(u_0), u_0 \in A, c^0(u_0) = 0$ for $u_0 \notin A$. Then define a 1-cocycle $z^1 \in Z^1(X, A; \Pi)$ by setting for any singular 1-simplex $u_1 \in X$

$$z_1(u_1) = -c^0(u_1^{(1)}) + c^0(u_1^{(0)})$$

the class of this z^1 is $\delta(h^0)$. As [8] shows the cohomology axioms of Eilenberg-Stienrod all holds for 0 and 1 dimensional cohomology with non-abelian coefficient in arbitrary spaces ,using continuous or singular maps.

2 Reduced free product

In this section we define the reduced free product of two topological groups and will find an exact sequence involving the cohomology groups in dimensions 0 and 1. The idea is motivated by [8].

Let G_1 and G_2 be topological groups. By $G_1 * G_2$ we mean the free topological product in the sense of [7].The concept is the natural analogue of a free product of groups[4] and was ispired by the work of Golema[2] and Hulanicki[5] on free product in the category of compact Hausdorff groups .

Definition 2.1 Let G_1, G_2 be topological groups Then $F = G_1 * G_2$ is said to be a free topological product if

(a) G_1 and G_2 are subgroups of F

(b) F is generated algebraically by $G_1 \cup G_2$.

(c) for each $i = 1, 2$,ϕ_i is a continuous homomorphism of G_i into a topological group H , then there exists a continuous homomorphism ϕ of F into H such that $\phi = \phi_i$ on G_i for each i.

By [7] it always exists , is unique and algebraically isomorphic to the usual free product of underlying groups.

Let $\Im$ be a system of groups and continuous homomorphisms:

$$(2.2) \qquad \Im : G_0 \begin{array}{l} \overset{\theta_1}{\nearrow} \; G_1 \\ \\ \underset{}{\overset{\theta_1}{\searrow}} \; G_2 \end{array}$$

Definition 2.3 .Let N be the closure of normal subgroup of $G_1 * G_2$ containing all elements of the form $\theta_1(g)\theta_2^{-1}(g)$ for all $g \in G_0$. Then $G_1 * G_2/N$ is called the reduced free product of G_1 and G_2.

There is another description of the free product as follows:

Definition 2.4 Let G be a topological group. We define a homomorphism $\Omega : \Im \to G$ to be a pair of homomorphisms $\Omega = (w_1, w_2)$ such that the following diagram is commutative:

$$(2.5) \qquad \begin{array}{ccccc} & & G_1 & & \\ & \overset{\theta_1}{\nearrow} & & \overset{w_1}{\searrow} & \\ G_0 & & & & G \\ & \overset{\theta_1}{\searrow} & & \overset{w_1}{\nearrow} & \\ & & G_2 & & \end{array}$$

(G, Ω) is called the direct limit of this system provided :

(i) Ω is onto i.e., the images of w_1, w_2 generate G

(ii) every homomorphism $\overline{\Omega} : \Im \to \Pi$ is covered by $\Omega : \Im \to G$ i.e., there exists a continuous homomorphism $h : G \to \Pi$ such that $\overline{w_i} = h \circ w_i, i = 0, 1$

With in isomorphism this direct limit is unique[8] and it is the reduced free product of $\Im$ defined above taking for Ω the homomorphism induced by the cononical imbeddings of G_1 and G_2 in $G_1 * G_2$.

An exact sequence:

Given any topological group Π , the diagram (2.5) induces a commutative diagram

$$
(2.6) \qquad 0 \to Hom(G,\Pi) \begin{array}{c} \overset{w_1^*}{\nearrow} \\ \\ \underset{w_2^*}{\searrow} \end{array} \begin{array}{c} Hom(G_1,\Pi) \\ \\ \\ Hom(G_2,\Pi) \end{array} \begin{array}{c} \overset{\theta_1^*}{\searrow} \\ \\ \underset{\theta_2^*}{\nearrow} \end{array} Hom(G_0,\Pi)
$$

This is exact if $\theta_1^*(h_1) = \theta_2^*(h_2)$ implies that there is a unique $h \in Hom(G,\Pi)$ such that $h_1 = w_1^*(h)), h_2 = w_2^*(h)$.

Lemma 2.7 (G,Π) *is the direct limit of* $\Im$ *if and only if (2.6) is exact.*

Proof. Let (G,Π) be the direct limit of $\Im$. If $\theta_1^*(h_1) = \theta_2^*(h_2)$, then (h_1,h_2) is a homomorphism $\Im \to \Pi$ and part (ii) of definition 2.5, guaranties the exactness of $h : G \to \Pi$ such that $h_1 = w_1^*(h)$, $h_2 = w_2^*(h)$. By (ii) h is unique. So (2.6) is exact.

Conversely if (2.6) is exact ,(ii) is immediate. Now we show part (i). Let G' be a subgroup of G generated by images of w_1,w_2 and let $w_1' : G_1 \to G_1'$, $w_2' : G_2 \to G'$ be the restriction of w_1, w_2. If $\Pi = G$ in (2.6) , then the exactness shows that there is a homomorphism $\alpha : G \to G'$ such that $w_1^*(\alpha) = w_1'$,$w_2'(\alpha) = w_2'$. Let $i : G' \to G$ be the injection . Now by (2.6) and $\Pi = G$ we have $w_1^*(i \circ \alpha) = w_1$, $w_2^*(i \circ \alpha) = w_2$. Since $w_1^*(1) = w_1$, $w_2^*(1) = w_2$, the exactness requires $i \circ \alpha = 1$ and hence $G' = G$.

Theorem 2.8 *If* A *,*B *and* $A \cap B$ *are path connected* $x_0 \in A \cap B$ *,* $A \cup B$ *metrizable and the injection*

$$
(2.9) \qquad k^* : H^1(A \cup B, x_0; \Pi) \sim H^1(S(A), S(B); \Pi)
$$

is an isomorphism onto ,then $\pi_1(A \cup B, x_0)$ *is the rdeuced free product of the system*

$$
(2.10) \qquad \begin{array}{ccc}
 & & \pi_1(A, x_0) \\
 & \overset{j_1}{\nearrow} & \\
\pi_1(A \cap B, x_0) & & \\
 & \overset{j_2}{\searrow} & \\
 & & \pi_1(B, x_0)
\end{array}
$$

j_1, j_2 being injections.

Proof. Denote by Δ' the composition $l_2^*(l_1^*)^{-1}\delta$ in

$$
\begin{array}{ccc}
H^0(A \cap B, x_0; \Pi) & & H^1(S(A) \cup S(B), S(B); \Pi) \\
\downarrow \delta & & \uparrow l_2^* \\
H^1(A, A \cap B; \Pi) & \overset{l_1^*}{\leftarrow} & H^1(S(A) \cup S(B), S(B); \Pi)
\end{array}
$$

where l_1^*,l_2^* are injections and δ is defined in (1.3). It is clear that l_1^* is an isomorphism [8] ,then $\Delta = (k^*)^{-1}\Delta' : H^0(A \cap B, x_0; \Pi) \to H^1(A \cup B, x_0; \Pi)$

Therefore , we have the following diagram

$$
(2.11) \qquad .. \to H^0(A \cap B, x_0; \Pi)
$$

$$
\begin{array}{ccccc}
 & & H^1(A, x_0; \Pi) & & \\
 & \overset{i_1^*}{\nearrow} & & \overset{j_1^*}{\searrow} & \\
\overset{\Delta}{\to} H^1(A \cup B, x_0; \Pi) & & & & H^1(A \cap B, x_0; \Pi) \\
 & \overset{i_2^*}{\searrow} & & \overset{j_2^*}{\nearrow} & \\
 & & H^1(B, x_0; \Pi) & &
\end{array}
$$

Since k^* is an isomorphism so (2.11) is well-defined .

By [8,theorem 1], $H^0(A \cap B, x_0; \Pi) = 0$ since $A \cap B$ is path connected. Now by lemma 2.7, the result follows.

References

[1] R. C. Alperin,H. Sahleh, Hopf,s Formula and the Schur multiplicator for topological groups,*Kyungpook .Math.Journal*, **31**(1)(1991), 35-71.

[2] D.K. Biss , The topological fundamental group and generalized covering spaces,*Topology .Appl*, **124**(3) (2002), 355-371.

[3] K.Golema, Free product of compact general algebras,*Colloq.Math*, **13**(1965), 165-166.

[4] M.Hall , The theory of groups ,the Macmillan Company,New York, 1959.

[5] A.Hulanicki, Isomorphic imbeddingof free products of compact groups , *Colloq.Math*, **16** (1967),235-241.

[6] A.A.Markov, On free topological groups,*Amer.Math.Soc.Transl*, 30(1950), 11-88.

[7] S.A. Morris, Free product of Topological groups, *Bull.Austral.Math.Soc*, **4** (1971), 17-29.

[8] P.Olum, Non-abelian cohomology and Van Kampen,s theorem, *Annl.Math* , **68**(3) (1958) , 658-668.

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