Construction of singular hypersurfaces and linkage over a finite field

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Abstract. Here we prove two existence theorem over $\mathbb{F}_q$: existence of hypersurfaces with prescribed isolated singularities and existence of “smooth” linkage.

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1. The statements

Here we consider two existence theorems over $\mathbb{F}_q$. The corresponding constructions are obvious over $\bar{\mathbb{F}}_q$ and the aim is just to find a relatively low prime power $q$ such that the same constructions may be done over $\mathbb{F}_q$. For any $P \in \mathbb{P}^n(\bar{\mathbb{F}}_q)$ and any integer $m > 0$ let $mP$ denote the infinitesimal neighborhood of order $m - 1$ of $P$ in $\mathbb{P}^n$. Set $0P = \emptyset$. In section 2 we will study the case of hypersurfaces with prescribed isolated singularities and prove the following result.

Theorem 1. Fix a prime power $q$, an integer $n \geq 2$, an integer $d > 0$, an integer $s$ such that $1 \leq s \leq (q^{n+1} - 1)/(q - 1)$, integers $m_i > 0$, and $s$ distinct points $P_1, \ldots, P_s \in \mathbb{P}^n(\mathbb{F}_q)$. Let $Z := \bigcup_{i=1}^s m_i P_i$ and assume $h^1(\mathbb{P}^n, \mathcal{I}_Z(d - 1)) = 0$. Set $\delta := dn - \sum_{i=1}^s m_i^{n+1}$ and $\delta_i := m_i^{n+1}$. Assume $q \geq (\delta - 1)\delta^n$. Then there exists a degree $d$ hypersurface $X \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ and such that $\text{Sing}(X) \subseteq \{P_1, \ldots, P_s\}$, $P_i \in \text{Sing}(X)$ if and only if $m_i \geq 2$, and $X$ has multiplicity $m_i$ at each $P_i$. Furthermore, if $q \geq (\delta - 1)\delta^n + \sum_{i=1}^s (\delta_i - 1)\delta_i^{n-1}$, then we may find $X$ such that $X$ has an ordinary multiple point with multiplicity $m_i$ at $P_i$, i.e. the tangent cone of $X$ at $P_i$ is a cone over a smooth degree $m_i$ hypersurface of $\mathbb{P}^{n-1}$.

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When \( P_1, \ldots, P_s \in \mathbb{P}^n(\overline{\mathbb{F}}_q) \), \( P_i \notin \mathbb{P}^n(\mathbb{F}_q) \) for some \( i \), but the set of all pairs \( \{(P_1, m_1), \ldots, (P_s, m_s)\} \) is invariant for the natural action of the absolute Galois group of \( \mathbb{F}_q \) we are able to prove the following result.

**Theorem 2.** Fix a prime power \( q \), an integer \( n \geq 2 \), an integer \( d > 0 \), an integer \( s \) such that \( 1 \leq s \leq (q^{n+1} - 1)/(q - 1) \), integers \( m_i > 0 \), and \( s \) distinct points \( P_1, \ldots, P_s \in \mathbb{P}^n(\overline{\mathbb{F}}_q) \). Let \( Z := \bigcup_{i=1}^s m_i P_i \) and assume \( h^1(\mathbb{P}^n, \mathcal{I}_Z(d - 1)) = 0 \). Assume that the scheme \( Z \) and the inclusion of \( Z \) in \( \mathbb{P}O^n \) are defined over \( \mathbb{F}_q \), i.e. assume that the absolute Galois group of \( \mathbb{F}_q \) acts trivially on the set of pairs \( \{(P_1, m_1), \ldots, (P_s, m_s)\} \). Set \( \delta := d^n - \sum_{i=1}^s m_i^n \) and \( \delta_i := m_i^{n-1} \). Assume \( q \geq (\delta - 1)\delta^n \). Then there exists a degree \( d \) hypersurface \( X \subset \mathbb{P}^n \) defined over \( \mathbb{F}_q \) and such that \( \text{Sing}(X) \subseteq \{P_1, \ldots, P_s\} \), \( P_i \in \text{Sing}(X) \) if and only if \( m_i \geq 2 \), and \( X \) has multiplicity \( m_i \) at each \( P_i \). Furthermore, if \( q \geq (\delta - 1)\delta^n + \sum_{i=1}^s (\delta_i - 1)\delta_i^{n-1} \), then we may find \( X \) such that \( X \) has an ordinary multiple point with multiplicity \( m_i \) at \( P_i \), i.e. the tangent cone of \( X \) at \( P_i \) is a cone over a smooth degree \( m_i \) hypersurface of \( \mathbb{P}^{n-1} \).

**Remark 1.** Take \( Z \) as in the statements of Theorems 1 and 2 and let \( \mu \) be the first integer \( t \geq -1 \) such that \( h^1(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0 \). Thus \( h^1(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0 \) for all \( t \geq \mu \) and \( d \geq \mu + 1 \). It is classical that \( \mu \leq m_1 + \cdots + m_s - 1 \) and that we have equality if and only if the points \( P_1, \ldots, P_s \) are collinear ([3]). If the points \( P_1, \ldots, P_s \) are in linearly general position and \( m_1 \geq m_2 \geq \cdots \geq m_s \), then \( \mu \leq \max\{m_1 + m_2 - 1, (m_1 + \cdots + m_s + n - 2)/n\} \) ([3]).

Then we will consider a problem of “nice” linkage over \( \mathbb{F}_q \) (see [2] for general theory).

**Theorem 3.** Fix integers \( n \geq r \geq 2 \) and a prime power \( q \). Let \( C \subset \mathbb{P}^n \) a smooth subscheme with pure codimension \( r \) defined over \( \mathbb{F}_q \). Let \( \mu \) be the first non-negative integer \( z \) such that \( h^i(\mathbb{P}^n, \mathcal{I}_C(z - i)) = 0 \) for all \( i \geq 1 \). Fix \( r \) integers \( t_1 \geq \cdots \geq t_r \geq \mu + 1 \). Assume \( q \geq \sum_{i=1}^r (t_i^n - 1)t_i^2 \). Then there are degree \( t_i \) hypersurfaces \( A_i \subset \mathbb{P}^n \) defined over \( \mathbb{F}_q \) such that \( A_1 \cap \cdots \cap A_r \) is a codimension \( r \) hypersurface containing \( C \), reduced along \( C \) and smooth outside \( C \).

In the statement of Theorem 3 we do not assume that \( C \) is connected or that it is geometrically connected. If \( C \) is not geometrically connected we do not assume that all the irreducible components of \( C(\mathbb{F}_q) \) are defined over \( \mathbb{F}_q \).

2. The proofs

*Proof of Theorem 1.* Since \( \dim(Z) = 0 \) we have \( h^j(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0 \) for all \( t \in \mathbb{Z} \) and all \( j \) such that either \( j \geq 2 \) and \( t \geq -n \) or \( 2 \leq j \leq n-1 \). Let \( \mu \) be the first integer \( t \geq -1 \) such that \( h^1(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0 \). Thus \( h^1(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0 \) for all \( t \geq \mu \) and \( d \geq \mu + 1 \). By Castelnuovo-Mumford’s lemma the homogeneous ideal of \( Z \) is generated by forms of degree at most \( \mu + 1 \) and hence it is generated by forms of degree at most \( d \). Let \( v : M \to \mathbb{P}^n \) be the blowing-up of \( \mathbb{P}^n \) at
the points $P_1, \ldots, P_s$. We have $R^j_i(\mathcal{O}_M) = 0$ for all $j \geq 1$ and $v_*(\mathcal{O}_M) = \mathcal{O}_{\mathbb{P}^s}$.

Set $E_i := v^{-1}(P_i)$. Hence $E_i$, $1 \leq i \leq s$. Hence $\text{Pic}(M) \cong \mathbb{Z}^{s+1}$ and $\text{Pic}(M)$ is freely generated by the classes of the line bundles $v^*(\mathcal{O}_{\mathbb{P}^s}(1))$ and $\mathcal{O}_M(E_i)$, $1 \leq i \leq s$. For all integers $t, z, z_i$, $1 \leq i \leq s$, set $\mathcal{L}_{t,z} := v^*(\mathcal{O}_{\mathbb{P}^s}(t))(-z_1 E_1 - \cdots - z_s E_s)$ and $\mathcal{L}_{t,z_1,\ldots,z_s} := v^*(\mathcal{O}_{\mathbb{P}^s}(t))(-z_1 E_1 - \cdots - z_s E_s)$.

Since $P_i \in \mathbb{P}^n(\mathbb{F}_q)$ for all $i, v, M$, each $E_i$ and all $\mathcal{L}_{t,z}$ and $\mathcal{L}_{t,z_1,\ldots,z_s}$ are defined over $\mathbb{F}_q$. If $z_i \geq 0$ for all $i$, then $v_*(\mathcal{L}_{t,z_1,\ldots,z_s}) = \mathcal{I}_{v_{i=1}^s z_i P_i}(t)$.

(a) Here we will check that $R^j_i(\mathcal{L}_{t,z_1,\ldots,z_s}) = 0$ for all integers $j, t, z_1, \ldots, z_s$ such that $j \geq 1$ and $z_i \geq 0$ for all $i$. By the projection formula it is sufficient to prove the case $t = 0$. The result is true if $z_i = 0$ for all $i$. Hence we may assume $z_i > 0$ for some $i$ and use induction on the integer $z_1 + \cdots + z_s$. Hence we may assume that the result is true for the integers $z_1, \ldots, z_{i-1}, z_i - 1, z_{i+1}, \ldots, z_s$.

Set $B := \cup_{i=1}^s z_i E_i$ and $B' := B - E_i$. Thus we have the following exact sequence on $M$:

$$0 \to \mathcal{I}_B \to \mathcal{I}_{B'} \to \mathcal{O}_{E_i}(B') \to 0$$

Apply the direct image functor to (1), the cohomology of $E_i \cong \mathbb{P}^{n-1}$ and that $\mathcal{O}_{E_i}(B')$ is a degree $z_i - 1$ line bundle on $E_i$.

(b) By part (a) and the definition of $\mu$ we have $h^j(M, \mathcal{L}_{t,m_1,\ldots,m_s}) = 0$ and $h^0(M, \mathcal{L}_{t,m_1,\ldots,m_s}) = \binom{n+s}{n} - \sum_{i=1}^s \binom{m_i + n - 1}{n - 1}$ for all $j \geq 1$, and $t \geq \mu$ and in particular for all $j \geq 1$ and $t \geq 1$. In the same way we get that $h^1(M, \mathcal{L}_{t,z_1,\ldots,z_s}(-E_i)) = 0$ for all $t \geq \mu + 1$.

(c) Here we will show that $\mathcal{L}_{t,m_1,\ldots,m_s}$ is very ample for all $t \geq \mu + 1$ and in particular for $t = d$. It is sufficient to show the surjectivity of the restriction map $\rho_{A,t} : H^0(M, \mathcal{L}_{t,m_1,\ldots,m_s}) \to H^0(A, \mathcal{L}_{t,m_1,\ldots,m_s})$ for all zero-dimensional subschemes $A \subset M$ such that $\text{length}(A) = 2$. We distinguish six cases.

(i) $A$ is reduced, say $A = \{Q, Q'\}$ with $Q \neq Q'$, and $A \cap (E_1 \cup \cdots \cup E_s) = \emptyset$;

(ii) $A$ is not reduced and $Q := A_{\text{red}} \notin E_1 \cup \cdots \cup E_s$;

(iii) $A$ is reduced, say $A = \{Q, Q'\}$ with $Q \neq Q'$, $Q \in E_i$, $Q \in E_j$ and $i \neq j$;

(iv) $A$ is reduced, say $A = \{Q, Q'\}$ with $Q \neq Q'$, with $Q \in E_i$ and $Q' \notin E_1 \cup \cdots \cup E_s$;

(v) $A$ is not reduced, $Q := A_{\text{red}} \in E_i$, and $A$ is not contained in $E_i$;

(vi) $A \subset E_i$ for some $i$.

In cases (i), (ii), (iii), (iv), (v) the morphism $v|A : A \to \mathbb{P}^n$ is an embedding. In all these cases it is sufficient to use that the homogeneous ideal of $Z$ is generated by forms of degree at most $t$. Now assume that we are in case (vi).

We have $h^1(\mathbb{P}^n, \mathcal{I}_Z(t-1)) = 0$ for all schemes $Z' \subset Z$. Take the set-up of part (a) with respect to the integers $z_j := m_j$ for all $j$. Apply the twist by $\mathcal{L}_{t,0,\ldots,0}$ to the exact sequence (1), use the last vanishing of part (b) and that the line bundle $\mathcal{L}_{d,m_1,\ldots,m_s} | E_i$ is the degree $m_i$ line bundle on $E_i \cong \mathbb{P}^{n-1}$ and hence it is very ample.

(d) By part (c) the line bundle $\mathcal{L}_{d,m_1,\ldots,m_s}$ is very ample. Notice that we have $\deg(\mathcal{L}_{d,m_1,\ldots,m_s}) = d^n - \sum_{j=1}^s m_i^n = \delta$. By [1], Th. 1, there is a smooth
$W \in |\mathcal{L}_{d,m_1,...,m_s}|$. Set $X := v(W)$. Now we consider the "Furthermore" part. We need to find $W$ as above with the additional property that $W$ is transversal to each $E_i$. Since $\deg(\mathcal{L}_{d,m_1,...,m_s} \cap E_i) = m_i$, $\mathcal{L}_{d,m_1,...,m_s}$ embeds $E_i \cong \mathbb{P}^{n-1}$ by a subsystem of the degree $m_i$ Verone embedding. Hence the embedded projective space has degree $m_i^{n-1} = \delta_i$. Thus its dual variety $\Delta_i$ in the projective space $|\mathcal{L}_{d,m_1,...,m_s}|$ has degree at most $(\delta_i)^{n-1}$. The proof of [1], Lemma 1, and our assumption on $q$ implies the existence of a hyperplane of $|\mathcal{L}_{d,m_1,...,m_s}|$ transversal to the image of $M$ and to the images of all $E_i$.

*Proof of Theorem 2.* We use the set-up introduced in the proof of Theorem 1. Now some of the line bundles $\mathcal{O}_M(E_i)$ may not be defined over $\mathbb{F}_q$, but $v$, $M$ and all line bundles $\mathcal{L}_{t,z}$ are defined over $\bar{\mathbb{F}}_q$. Furthermore for any $t \in \mathbb{Z}$ the line bundle $\mathcal{L}_{t,m_1,...,m_s}$ is defined over $\mathbb{F}_q$. Working over $\bar{\mathbb{F}}_q$ the proof of Theorem 1 show that $\mathcal{L}_{d,m_1,...,m_s}$ is very ample. Hence we may again apply [1], Th. 1.

*Proof of Theorem 3.* Let $v : M \rightarrow \mathbb{P}^n$ be the blowing-up of $C$. Since $C$ is smooth, $M$ is smooth. Set $E := v^{-1}(C)$. For all integers $t$ set $\mathcal{L}_t := v^*((\mathcal{O}_{\mathbb{P}^n}(t)(-E))$. As in the proof of Theorem 1 it is easy to check that $\mathcal{L}_t$ is very ample for all $t \geq \mu = 1$. we again apply [1], Th. 1. Since a complete intersection in a smooth ambient has no embedded point, to check the existence of $X$ which is reduced along $C$ it is sufficient to test finitely many points of $C(\bar{\mathbb{F}}_q)$.

*References*


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