Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 8, 369 - 374

Existence of three solutions for a one-dimensional Dirichlet problem

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Abstract

In this paper, the existence of at least three weak solutions for Dirichlet problem

$$\begin{cases} -u'' + \lambda f(x,u) = 0, & x \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ and $f : [0,1] \times R \to R$ is a continuous function, is established. The approach is based on variational methods and critical points.

Mathematics Subject Classification: 35J20; 34A15

Keywords: Three solutions, Critical point, Multiplicity results, Dirichlet problem

1 Introduction

In this work, we study the boundary value problem

$$\begin{cases} -u'' + \lambda f(x,u) = 0, & x \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $\lambda > 0$ and $f : [0,1] \times R \to R$ is a continuous function. In this paper, under novel assumptions, we are interested in ensuring the existence of at least three weak solutions for the problem (1). As usual, a weak solution of (1) is any $u \in W_0^{1,2}([0,1])$ such that

$$\int_0^1 u'(x)v'(x)dx + \lambda \int_0^1 f(x,u(x))v(x)dx = 0$$

for all $v \in W_0^{1,2}([0,1])$.

Multiplicity results for the problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{2}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $f : \Omega \times R \to R$ is a Carathéodory function (see, for example, [1]) and in the case $N = 1,\ p = 2$ (see, for example, [3]) have been broadly investigated in recent years; for instance, in [1], using variational methods, the authors ensure the existence of a sequence of arbitrarily small positive solutions for problem (2) when the function f has a suitable oscillating behaviour at zero.

Also, in [3], the author proves multiplicity results for the problem

$$\begin{cases} u'' + \lambda f(u) = 0 \\ u(0) = u(1) = 0 \end{cases} \tag{3}$$

which for each $\lambda \in [0, +\infty[$, admits at least three solutions in $W_0^{1,2}([0,1])$ when f is continuous function.

In the present paper, our approach is based on a three critical points theorem proved in [4], recalled below for the reader's convenience (Theorem 1.1), on a technical lemma (Lemma 2.1) that allow us to apply it. Theorem 2.2 which is our main result, ensures the existence of an open interval $\Lambda \subseteq [0, \infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three weak solutions whose norms in $W_0^{1,2}([0,1])$ are less than q.

As a consequence of Theorem 2.2, we obtain Theorem 2.3 and, in turn, Corollary 2.4.

Theorem 2.3 ensures the existence of weak solutions for the boundary value problem

$$\begin{cases} -u'' + \lambda f(u(x)) = 0, & x \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

We here recall for the reader's convenience the three critical points theorem:

Theorem 1.1 : [Theorem 1 0f [4], with choosing $h(\lambda) = \rho\lambda$] Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a continuously *Gâteaux* differentiable and sequentially weakly lower semicontinuous functional whose

Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow R$ a continuously *Gâteaux* differentiable functional whose *Gâteaux* derivative is compact. Assume that

$$\lim_{||u||\to+\infty}(\Phi(u)+\lambda\Psi(u))=+\infty$$

for all $\lambda \in [0,+\infty[$, and that there exists $\rho \in R$ such that

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\rho\lambda).$$

Then, there exists an open interval $\Lambda \subseteq [0,+\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u)+\lambda\Psi'(u)=0$$

has at least three solutions in X whose norms are less than q.

In Proposition 2.2 of [2] if $-J=\Psi$. Then we have:

Proposition 1.2: Let X be a non-empty set, and $\Phi,\ \Psi$ two real functions on X. Assume that there are $r \in R,\ u_0,\ u_1 \in X$ such that $\Phi(u_0)=\Psi(u_0)=0$, $\Phi(u_1)>r$, $\inf_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) > r\frac{\Psi(u_1)}{\Phi(u_1)}$. Then, for each ρ satisfying

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < \rho < r\frac{(-\Psi(u_1))}{\Phi(u_1)}$$

one has

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\rho\lambda).$$

2 Main Results

Here and in the sequel, X will denote the Sobolev space $W_0^{1,2}([0,1])$ with the norm $\| u \| = \left(\int_0^1 |u'(x)|^2dx\right)^{1/2}$ and put

$$g(x,t)=\int_0^t f(x,\xi)d\xi$$

for each $(x,t) \in [0,1]\times R$. Our main results fully depend on the following lemma:

Lemma 2.1 *Assume that there exist two positive constants c and d with* $c < \frac{d}{\sqrt{2}}$, *such that*

(i) $g(x,t) \leq 0$ *for each* $(x,t) \in [0,\frac{1}{2}]\times[0,d]$, *(ii)* $\frac{1}{2c^2}\min_{(x,t)\in[0,1]\times[-c,c]} g(x,t) > \frac{1}{d^2}\int_{\frac{1}{2}}^{1} g(x,d)dx.$

Then, there exist $r>0$ *and* $w\in X$ *such that* $||w||^2>2r$ *and*

$$\min_{(x,t)\in[0,1]\times[-\sqrt{r/2},\sqrt{r/2}]} g(x,t) > 2r\frac{\int_0^1 g(x,w(x))dx}{||w||^2}.$$

Proof: *We put*

$$w(x)=\begin{cases} 2dx & 0\le x\le \frac{1}{2},\\ d & \frac{1}{2}\le x\le 1,\end{cases}$$

and $r=2c^2$. *It is easy to see that* $w\in X$ *and, in particular, one has*

$$||w||^2=2d^2.$$

Hence, taking into account that $c<\frac{d}{\sqrt{2}}$, *one has* $2r<||w||^2$.
Since $0\le w(x)\le d$ *for each* $x\in[0,1]$, *condition (i) ensures that*

$$\int_0^{\frac{1}{2}} g(x,w(x))dx \le 0. \tag{5}$$

Moreover, from (ii) and (5), *we have*

$$\min_{(x,t)\in[0,1]\times[-\sqrt{r/2},\sqrt{r/2}]} g(x,t) > (\frac{\sqrt{2}c}{d})^2\int_{\frac{1}{2}}^{1} g(x,d)dx \ge 2r\frac{\int_0^1 g(x,w(x))dx}{||w||^2}.$$

Namely

$$\min_{(x,t)\in[0,1]\times[-\sqrt{r/2},\sqrt{r/2}]} g(x,t) > 2r\frac{\int_0^1 g(x,w(x))dx}{||w||^2}.$$

So, the proof is complete.

Now, we state our main result:

Theorem 2.2 *Assume that there exist* $a(x)\in C([0,1])$ *which* $a(x)\le 0$ *on* $[0,\frac{1}{2}]$ *and three positive constants* c, d, s *with* $c<\frac{d}{\sqrt{2}}$ *and* $s<2$ *such that*

(i) $g(x,t)\le 0$ *for each* $(x,t)\in[0,\frac{1}{2}]\times[0,d]$,
(ii) $\frac{1}{2c^2}\min_{(x,t)\in[0,1]\times[-c,c]} g(x,t) > \frac{1}{d^2}\int_{\frac{1}{2}}^{1} g(x,d)dx.$
(iii) $g(x,t)\ge a(x)(1+|t|^s)$ *almost everywhere in* $[0,1]$ *and for each* $t\in R$.

Then, there exists an open interval $\Lambda\subseteq[0,+\infty[$ *and a positive real number* q *such that, for each* $\lambda\in\Lambda$, *problem* (1) *admits at least three solutions in* X *whose norms are less than* q.

Proof: *For each $u \in X$, we put*

$$\Phi(u) = \frac{||u||^2}{2},$$

$$\Psi(u) = \int_0^1 g(x, u(x))dx.$$

Of course, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. In particular, for each $u, v \in X$ one has

$$\Phi'(u)(v) = \int_0^1 u'(x)v'(x)dx,$$

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)dx.$$

Hence, the weak solutions of (1) *are exactly the solutions of the equation*

$$\Phi'(u) + \lambda\Psi'(u) = 0.$$

Thanks to (iii), for each $\lambda > 0$ one has that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty.$$

We claim that there exist $r > 0$ and $w \in X$ such that

$$\inf_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) > r\frac{\Psi(w)}{\Phi(w)}.$$

Now, taking into account that for every $u \in X$, one has

$$\max_{x \in [0,1]} |u(x)| \leq \frac{1}{2}||u||$$

for each $u \in X$, it follows that

$$\inf_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) = \inf_{||u||^2 \leq 2r} \int_0^1 g(x, u(x))dx \geq \min_{(x,t) \in [0,1] \times [-\sqrt{r/2}, \sqrt{r/2}]} g(x, t).$$

Now, thanks to Lemma 2.1, there exist $r > 0$ and $w \in X$ such that $||w||^2 > 2r$ and

$$\min_{(x,t) \in [0,1] \times [-\sqrt{r/2}, \sqrt{r/2}]} g(x, t) > 2r\frac{\int_0^1 g(x, w(x))dx}{||w||^2},$$

so, our claim is true. Therefore, using Proposition 1.2 with choose $u_1 = w$, there exists $\rho \in R$, such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, our conclusion follows from Theorem 1.1.

We now want to point out a consequence of Theorem 2.2:

Theorem 2.3 *Let $f : R \to R$ be a continuous function. Put $g(t) = \int_0^t f(\xi)d\xi$ for each $t \in R$ and assume that there exist three positive constants c, d, s with $c < \frac{d}{\sqrt{2}}$, $s < 2$ and negative constant η such that*

(j) $g(t) \leq 0$ for each $t \in [0, d]$,
(jj) $\frac{1}{c^2} \min_{t \in [-c,c]} g(t) > \frac{1}{d^2} g(d)$,
(jjj) $g(t) \geq \eta(1 + |t|^s)$ for each $t \in R$.

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (4) admits at least three solutions in X whose norms are less than q.

If $f(t) \leq 0$ for each $t \in [-c, \max\{c, d\}]$. Then, by using of the Theorem 2.3, we have the following result:

Corollary 2.4 *Let $f : R \to R$ be a continuous function. Put $g(t) = \int_0^t f(\xi)d\xi$ for each $t \in R$ and assume that there exist three positive constants c, d, s with $c < \frac{d}{\sqrt{2}}$, $s < 2$ and negative constant η such that*

(j′) $\frac{g(c)}{c^2} > \frac{g(d)}{d^2}$,
(jj′) $g(t) \geq \eta(1 + |t|^s)$ for each $t \in R$.

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (4) admits at least three solutions in X whose norms are less than q.

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Received: Dec. 27, 2005